The discriminant of a polynomial \( f = a \prod_{i=0}^{m}(x - \alpha_i) \) is defined by
\[
D(f) = a^{2m-2} \prod_{1 \leq i < j \leq m} (\alpha_j - \alpha_i)^2.
\]

The discriminant is a useful tool to get information about the roots of a high-degree polynomial.

In this paper, the author gives a lower bound for the discriminant of the polynomials \( \{G_{k,i}\} \) defined by
\[
G_{k,0}(x) = 1, \quad G_{k,1}(x) = x + 1 \quad \text{and then recursively}
\]
\[
G_{k,i+2}(x) = xG_{k,i+1}(x) - (k - 1)G_{k,i}(x).
\]

These polynomials have been used for the study of graphs and their girth (this is the length of the shortest circuit in the graph), specially due to its relation with the Moore Bound
\[
M_{d,k} = \begin{cases} 
1 + \frac{d^{d-1}k^{-1}-1}{d-2} & d > 2 \\
2k + 1 & d = 2 
\end{cases},
\]

which relates the degree, the order, the diameter and the girth of a graph. The discriminant of these polynomials have been used to prove the existence of graphs with certain degree or girth as in [C. Delorme and G. Pineda-Villavicencio, Electron. J. Comb. 17, No. 1, Research Paper R143, 25 p. (2010; Zbl 1204.05043)].

While all of this is addressed in the second section of the paper and the author suggests a path to proof his main result in a graph-theoretical way, most of the document is devoted to proof the result in a more elementary way. The main idea is to use orthogonality of some polynomials related to \( \{G_{k,i}\} \) and from there establish a relation through simple inequalities with the discriminant of \( G_{k,d} \). While the notation could be a little heavy, the results are quite simple to follow, leading to the following bound for the discriminant:
\[
D(G_{k,d}) > d^d(k-2) \left[ \sqrt{k(k-1)^2} - 2 \right]^{d-2}.
\]

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MSC:
- 05E30 Association schemes, strongly regular graphs
- 05C12 Distance in graphs
- 05C07 Vertex degrees
- 33C45 Orthogonal polynomials and functions of hypergeometric type (Jacobi, Laguerre, Hermite, Askey scheme, etc.)
- 42C05 Orthogonal functions and polynomials, general theory of nontrigonometric harmonic analysis

Keywords:
- discriminant; lower bound; Chebyshev polynomial; spectral excess theorem; Moore graphs; orthogonal polynomials

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References: