Cho, Eun-Kyung; Choi, Ilkyoo; Park, Boram

Partitioning planar graphs without 4-cycles and 5-cycles into bounded degree forests. (English) Zbl 1455.05059

Discrete Math. 344, No. 1, Article ID 112172, 9 p. (2021).

Summary: In 1976, Steinberg conjectured that planar graphs without 4-cycles and 5-cycles are 3-colorable. This conjecture attracted numerous researchers for about 40 years, until it was recently disproved by *V. Cohen-Addad* et al. [J. Comb. Theory, Ser. B 122, 452–456 (2017; Zbl 1350.05018)]. However, coloring planar graphs with restrictions on cycle lengths is still an active area of research, and the interest in this particular graph class remains.

Let G be a planar graph without 4-cycles and 5-cycles. For integers d_1 and d_2 satisfying $d_1 + d_2 \geq 8$ and $d_2 \geq d_1 \geq 2$, it is known that $V(G)$ can be partitioned into two sets V_1 and V_2, where each V_i induces a graph with maximum degree at most d_i. Since Steinberg's Conjecture is false, a partition of $V(G)$ into two sets, where one induces an empty graph and the other induces a forest is not guaranteed. Our main theorem is at the intersection of the two aforementioned research directions. We prove that $V(G)$ can be partitioned into two sets V_1 and V_2, where V_1 induces a forest with maximum degree at most 3 and V_2 induces a forest with maximum degree at most 4; this is both a relaxation of Steinberg's conjecture and a strengthening of results by *P. Sittitrai* and *K. Nakprasit* [Discrete Math. 341, No. 8, 2142–2150 (2018; Zbl 1388.05072)] in a much stronger form.

MSC:

05C70 Edge subsets with special properties (factorization, matching, partitioning, covering and packing, etc.)

05C10 Planar graphs; geometric and topological aspects of graph theory

Full Text: DOI

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