Choi, Jinwon; **van Garrel, Michel**; **Katz, Sheldon**; **Takahashi, Nobuyoshi**
Log BPS numbers of log Calabi-Yau surfaces. (English) Zbl 07288869
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Summary: Let (S, E) be a log Calabi-Yau surface pair with E a smooth divisor. We define new conjecturally integer-valued counts of $\mathbb{A}^1$-curves in (S, E). These log BPS numbers are derived from genus 0 log Gromov-Witten invariants of maximal tangency along E via a formula analogous to the multiple cover formula for disk counts. A conjectural relationship to genus 0 local BPS numbers is described and verified for del Pezzo surfaces and curve classes of arithmetic genus up to 2. We state a number of conjectures and provide computational evidence.

MSC:

14N35 Gromov-Witten invariants, quantum cohomology, Gopakumar-Vafa invariants, Donaldson-Thomas invariants (algebro-geometric aspects)
14J33 Mirror symmetry (algebro-geometric aspects)

Full Text: DOI

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