Jana, Nabin K.; Karn, Anil K.; Peralta, Antonio M.

Absolutely compatible pairs in a von Neumann algebra. (English) Zbl 1443.46035

Electron. J. Linear Algebra 35, 599-618 (2019).

Summary: Let a, b be elements in a unital C^*-algebra with $0 \leq a, b \leq I$. The element a is absolutely compatible with b if

$$|a-b| + |I-a-b| = I.$$

In this note, some technical characterizations of absolutely compatible pairs in an arbitrary von Neumann algebra are found. These characterizations are applied to measure how far are two absolute compatible positive elements in the closed unit ball from being mutually orthogonal or commuting. In the case of 2 by 2 matrices, the results admit a geometric interpretation. Namely, non-commutative matrices of the form $a = \begin{pmatrix} t & \alpha \\ \overline{\alpha} & 1-t \end{pmatrix}$ and $b = \begin{pmatrix} x & \beta \\ \overline{\beta} & 1-x \end{pmatrix}$ with $x, t \in (0,1)\{\frac{1}{2}\}$, $|\alpha|^2 < t(1-t)$ and $|\beta|^2 < x(1-x)$, are absolutely compatible if, and only if, the corresponding point $\widetilde{b} = (x, \mathrm{Re}(\beta), \mathrm{Im}(\beta))$ in $\mathbb{R}^3$ lies in the ellipsoid

$$\mathcal{E}a = \{\overline{x} \in \mathbb{R}^3 : d_2(\overline{x}, \widetilde{a}) + d_2(\overline{x}, \widetilde{a'}) = 1\},$$

where d_2 denotes the Euclidean distance in $\mathbb{R}^3$, and the elements $\widetilde{a}$ and $\widetilde{a'}$, are $(t, \mathrm{Re}(\alpha), \mathrm{Im}(\alpha))$ and $(1-t, -\mathrm{Re}(\alpha), -\mathrm{Im}(\alpha))$, respectively. The description of absolutely compatible pairs of positive 2 by 2 matrices is applied to determine absolutely compatible pairs of positive elements in the closed unit ball of $\mathbb{M}_n$.

MSC:

46L10 General theory of von Neumann algebras
46B40 Ordered normed spaces
46L05 General theory of C^*-algebras

Keywords:

absolute compatibility; commutativity; C^*-algebra; von Neumann algebra; projection; partial isometry; linear absolutely compatible preservers

Full Text: Link

References:

[1] H.F. Baker.Principles of Geometry, Vol. 3,Solid Geometry(reprint of the 1923 original). Cambridge Library Collection, Cambridge University Press, Cambridge, 2010.

[2] M. Burgos, F.J. Fern´andez-Polo, J.J. Garc´es, J. Mart´ınez Moreno, and A.M. Peralta.Orthogonality preservers in C∗-algebras, JB∗-algebras and JB∗-triples.J. Math. Anal. Appl., 348:220-233, 2008.

[3] N.K. Jana, A.K. Karn, and A.M. Peralta. Contractive linear preservers of absolutely compatible pairs betweenC∗algebras.Rev. R. Acad. Cienc. Exactas F´ıs. Nat. Ser. A Mat. RACSAM, 113:2731-2744, 2019. · Zbl 1434.46033

[4] R.V. Kadison and J.R. Ringrose.Fundamentals of the Theory of Operator Algebras, Vol. I,Elementary Theory. Academic Press, Inc., New York, 1983. · Zbl 0518.46046

[5] S. Kakutani. Concrete representation of abstract (M)-spaces.Ann. of Math. (2), 42:994-1024, 1941. · Zbl 0060.26604

[6] A.K. Karn. Ap-theory of ordered normed spaces.Positivity, 14:441-458, 2010. · Zbl 1225.46014

[7] A.K. Karn. Orthogonality in‘p-spaces and its bearing on ordered Banach spaces.Positivity, 18:223-234, 2014. · Zbl 1314.46026

[8] A.K. Karn. Orthogonality in aC∗-algebra.Positivity, 20:607-620, 2016. · Zbl 1361.46017

[9] A.K. Karn. Algebraic orthogonality and commuting projections in operator algebras.Acta Sci. Math. (Szeged), 84:323- 353, 2018. · Zbl 1413.46023

[10] J.-H. Liu, C.-Y. Chou, C.-J. Liao, and N.-C. Wong. Disjointness preservers ofAW∗-algebras.Linear Algebra Appl., 552:71-84, 2018. · Zbl 1398.46046

[11] J.-H. Liu, C.-Y. Chou, C.-J. Liao, and N.-C. Wong. Linear disjointness preservers of operator algebras and related structures.Acta Sci. Math. (Szeged), 84:277-307, 2018. · Zbl 1413.47063

[12] G.K. Pedersen.C∗-Algebras and their Automorphism Groups. London Mathematical Society Monographs, Vol. 14, Academic Press, Inc., London, 1979. · Zbl 0416.46043

[13] M.H. Stone. A general theory of spectra. I.Proc. Nat. Acad. Sci. U.S.A., 26:280-283, 1940. · Zbl 66.1284.03

[14] M. Wolff. Disjointness preserving operators inC∗-algebras.Arch. Math. (Basel), 62:248-253, 1994. · Zbl 0803.46069

[15] N.-C. Wong. Triple homomorphisms ofC∗-algebras.Southeast Asian Bull. Math., 29:401-407, 2005. · Zbl 1108.46041