Lercier, Reynald; **Ritzenthaler, Christophe**; **Sijsling, Jeroen**
Reconstructing plane quartics from their invariants. (English) Zbl 1439.13020
Discrete Comput. Geom. 63, No. 1, 73-113 (2020).

Let $R_{3,n}$ denote the ring of ternary forms of degree n over the complex field $\mathbb{C}$. Consider the action of $\mathrm{SL}_3(\mathbb{C})$ on the ring $R_{3,n}$. Explicit generators are known for $n \leq 4$. While the cases $n \leq 3$ are classically known, the case of $n = 4$ was shown by *J. Dixmier* [Adv. Math. 64, 279–304 (1987; Zbl 0668.14006)] and by Ohno (unpublished, see also *A.-S. Elsenhans* [J. Symb. Comput. 68, Part 2, 109–115 (2015; Zbl 1360.13017)] and *M. Girard* and *D. R. Kohel* [Lect. Notes Comput. Sci. 4076, 346–360 (2006; Zbl 1143.14304)]). By the work of these authors it follows that the ring $\mathbb{C}[R_{3,4}]^{\mathrm{SL}_3(\mathbb{C})}$ is generated by 13 elements, the so-called Dixmier-Ohno invariants of ternary quartics. The main result of the present paper is an explicit method that, given a generic tuple of Dixmier-Ohno invariants, reconstructs a corresponding plane quartic curve. The main technical tool is a method of *J.-F. Mestre* [Prog. Math. 94, 313–334 (1991; Zbl 0752.14027)], see also the authors in [Open Book Ser. 1, 463–486 (2013; Zbl 1344.11049)]. A Magma package of the authors for reconstructing plane quartics from Dixmier-Ohno invariants is available under `https://github.com/JRSijsling/quartic_reconstruction/`.

Reviewer: Peter Schenzel (Halle)

MSC:

13A50 Actions of groups on commutative rings; invariant theory
14L24 Geometric invariant theory
14H10 Families, moduli of curves (algebraic)
14H25 Arithmetic ground fields for curves

Keywords:

plane quartic curves; invariant theory; Dixmier-Ohno invariants; moduli spaces; reconstruction

Software:

GitHub; LiE; Magma; quartic_reconstruction

Full Text: DOI

References:

[1] Artebani, M.; Quispe, S., Fields of moduli and fields of definition of odd signature curves, Arch. Math. (Basel), 99, 4, 333-344 (2012) · Zbl 1267.14040
[2] Basson, R.: Arithmétique des espaces de modules des courbes hyperelliptiques de genre \(3\) en caractéristique positive. PhD thesis, Université de Rennes 1 (2015). https://tel.archives-ouvertes.fr/tel-01170922
[3] Böhning, Christian, The Rationality of the Moduli Space of Curves of Genus 3 after P. Katsylo, Cohomological and Geometric Approaches to Rationality Problems, 17-53 (2009), Boston: Birkhäuser Boston, Boston · Zbl 1197.14010
[4] Bosma, W.; Cannon, J.; Playoust, C., The Magma algebra system. I. The user language, J. Symb. Comput., 24, 3-4, 235-265 (1997) · Zbl 0898.68039
[5] Brouwer, Ae; Popoviciu, M., The invariants of the binary decimic, J. Symb. Comput., 45, 8, 837-843 (2010) · Zbl 1192.13005
[6] Brouwer, Ae; Popoviciu, M., The invariants of the binary nonic, J. Symb. Comput., 45, 6, 709-720 (2010) · Zbl 1189.13005
[7] Clebsch, A., Theorie der binären algebraischen Formen (1872), Leipzig: B.G. Teubner, Leipzig
[8] Cohen, H., Frey, G., Avanzi, R., Doche, C., Lange, T., Nguyen, K., Vercauteren, F. (eds.): Handbook of Elliptic and Hyperelliptic Curve Cryptography. Discrete Mathematics and its Applications (Boca Raton). Chapman \& Hall/CRC, Boca Raton (2006) · Zbl 1082.94001
[9] Dixmier, J., On the projective invariants of quartic plane curves, Adv. Math., 64, 3, 279-304 (1987) · Zbl 0668.14006
[10] Dolgachev, I., Lectures on Invariant Theory (2003), Cambridge: Cambridge University Press, Cambridge · Zbl 1023.13006
[11] Elsenhans, A-S, Explicit computations of invariants of plane quartic curves, J. Symb. Comput., 68, 2, 109-115 (2015) · Zbl 1360.13017

[12] Von Gall, Das vollständige Formensystem einer binären Form achter Ordnung, Mathematische Annalen, 17, 1, 31-51 (1880) · Zbl 12.0093.02

[13] Freiherr Von Gall, A., Das vollständige Formensystem der binären Form 7ter Ordnung, Math. Ann., 31, 3, 318-336 (1888) · Zbl 20.0128.01

[14] Fulton, W.; Harris, J., Representation Theory (1991), New York: Springer, New York

[15] Gatti, V.; Viniberghi, E., Spinors of \(13\)-dimensional space, Adv. Math., 30, 2, 137-155 (1978) · Zbl 0429.20043

[16] Girard, M.; Kohel, Dr; Hess, F.; Pauli, S.; Pohst, M., Classification of genus 3 curves in special strata of the moduli space, Algorithmic Number Theory, 346-360 (2006), Berlin: Springer, Berlin · Zbl 1143.14304

[17] Gordan, P., Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist, J. Reine Angew. Math., 69, 323-354 (1868) · Zbl 01.0046.01

[18] Hashimoto, M., Equivariant total ring of fractions and factoriality of rings generated by semi-invariants, Commun. Algebra, 43, 4, 1524-1562 (2015) · Zbl 1391.13013

[19] Hilbert, D.: Theory of Algebraic Invariants. Cambridge University Press, Cambridge: Translated from the German and with a preface by Reinhard C. Laubenbacher, Edited and with an introduction by Bernd Sturmfels (1993) · Zbl 0801.13001

[20] Katsylo, Pi; Vinberg, Eb, On the birational geometry of the space of ternary quartics, Lie Groups, Their Discrete Subgroups, and Invariant Theory, 95-103 (1992), Providence: American Mathematical Society, Providence · Zbl 0778.14006

[21] Katsylo, P., Rationality of the moduli variety of curves of genus 3, Comment. Math. Helv., 71, 4, 507-524 (1996) · Zbl 0885.14013

[22] Kılıçer, P., Labrande, H., Lercier, R., Ritzenthaler, C., Sijsling, J., Streng, M.: Plane quartics over \({\mathbb{Q}}\) with complex multiplication. Acta Arith. 185(2), 127-156 (2018). arXiv:1701.06489 · Zbl 1409.14051

[23] Kraft, H., Procesi, C.: Classical Invariant Theory. A Primer (1996). Notes available at https://www2.bc.edu/benjamin-howard/MATH8845/classical_invariant_theory.pdf

[24] Lercier, R., Lorenzo García, E., Ritzenthaler, C.: Reduction type of non-hyperelliptic genus 3 curves (2018). arXiv:1803.05816

[25] Lercier, R.; Ritzenthaler, C., Hyperelliptic curves and their invariants: geometric, arithmetic and algorithmic aspects, J. Algebra, 372, 595-636 (2012) · Zbl 1276.14088

[26] Lercier, Reynald; Ritzenthaler, Christophe; Sijsling, Jeroen, Fast computation of isomorphisms of hyperelliptic curves and explicit Galois descent, The Open Book Series, 1, 1, 463-486 (2013) · Zbl 1344.11049

[27] Lercier, Reynald; Ritzenthaler, Christophe; Rovetta, Florent; Sijsling, Jeroen, Parametrizing the moduli space of curves and applications to smooth plane quartics over finite fields, LMS Journal of Computation and Mathematics, 17, A, 128-147 (2014) · Zbl 1333.14060

[28] Lercier, R., Ritzenthaler, C., Sijsling, J.: \(quartic_-\) reconstruction: a Magma package for reconstructing plane quartics from Dixmier-Ohno invariants. https://github.com/JRSijsling/quartic_reconstruction/ (2016)

[29] Looijenga, E.; Keum, J.; Kondō, S., Invariants of quartic plane curves as automorphic forms, Algebraic Geometry, 107-120 (2007), Providence: American Mathematical Society, Providence · Zbl 1126.14034

[30] Lorenzo García, E., Twists of non-hyperelliptic curves, Rev. Mat. Iberoam., 33, 1, 169-182 (2017) · Zbl 1381.11050

[31] Maeda, T., On the invariant field of binary octavics, Hiroshima Math. J., 20, 3, 619-632 (1990) · Zbl 0741.14004

[32] Mestre, J-F; Mora, T.; Traverso, C., Construction de courbes de genre \(2\) à partir de leurs modules, Effective Methods in Algebraic Geometry, 313-334 (1991), Boston: Birkhäuser, Boston · Zbl 0752.14027

[33] Mumford, D.; Fogarty, J.; Kirwan, F., Geometric Invariant Theory (1994), Berlin: Springer, Berlin · Zbl 0797.14004

[34] Olive, M., About Gordan's algorithm for binary forms, Found. Comput. Math., 17, 6, 1407-1466 (2017) · Zbl 1408.13017

[35] Olver, P.J.: Classical Invariant Theory. London Mathematical Society Student Texts, vol. 44. Cambridge University Press, Cambridge (1999) · Zbl 0971.13004

[36] Popov, Vl, Stability of the action of an algebraic group on an algebraic variety, Izv. Akad. Nauk SSSR Ser. Mat., 36, 371-385 (1972) · Zbl 0232.14018

[37] Procesi, C.: Invariant Theory. Monografías del Instituto de Matemática y Ciencias Afines, vol. 2. Instituto de Matemática y Ciencias Afines, IMCA, Lima (1998) · Zbl 1159.13300

[38] Rökaeus, K.; Aubry, Y.; Ritzenthaler, C.; Zykin, A., Computer search for curves with many points among abelian covers of genus 2 curves, Arithmetic, Geometry, Cryptography and Coding Theory, 145-150 (2012), Providence: American Mathematical Society, Providence · Zbl 1317.11006

[39] Salmon, G., A treatise on the higher plane curves: intended as a sequel to "A treatise on conic sections" (1960), New York: Chelsea, New York · Zbl 05.0340.03

[40] Shioda, T., On the graded ring of invariants of binary octavics, Am. J. Math., 89, 4, 1022-1046 (1967) · Zbl 0188.53304

[41] Springer, Ta, Invariant Theory. Lecture Notes in Mathematics (1977), Berlin: Springer, Berlin

[42] Stoll, M., Reduction theory of point clusters in projective space, Groups Geom. Dyn., 5, 2, 553-565 (2011) · Zbl 1242.11027

[43] Sturmfels, B., Algorithms in Invariant Theory. Texts and Monographs in Symbolic Computation (2008), Vienna: Springer, Vienna

[44] Van Leeuwen, Maa; Cohen, Am; Lisser, B., LiE, A Package for Lie Group Computations (1997), Amsterdam: Computer Algebra Nederland, Amsterdam

[45] van Rijnswou, S.M.: Testing the Equivalence of Planar Curves. PhD thesis, Technische Universiteit Eindhoven, Eindhoven (2001). https://research.tue.nl/en/publications/testing-the-equivalence-of-planar-curves · Zbl 0982.13004

[46] Weil, A., The field of definition of a variety, Am. J. Math., 78, 509-524 (1956) · Zbl 0072.16001