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Analytic expressions of the solutions of advection-diffusion problems in one dimension with discontinuous coefficients. (English) [Zbl 07106896](#)

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65C05 Monte Carlo methods

82C31 Stochastic methods (Fokker-Planck, Langevin, etc.) applied to problems in time-dependent statistical mechanics

35R05 PDEs with low regular coefficients and/or low regular data

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References:

- [1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, 9th ed., Dover, New York, 1970.
- [2] T. Appuhamillage, V. Bokil, E. Thomann, E. Waymire, and B. Wood, *Occupation and local times for skew Brownian motion with applications to dispersion across an interface*, *Ann. Appl. Probab.*, 21 (2011), pp. 183–214, <https://doi.org/10.1214/10-AAP691>. · [Zbl 1226.60113](#)
- [3] T. Appuhamillage, V. Bokil, E. Thomann, E. Waymire, and B. Wood, *Corrections: Occupation and local times for skew Brownian motion with applications to dispersion across an interface*, *Ann. Appl. Probab.*, 21 (2011), pp. 2050–2051, <https://doi.org/10.1214/11-AAP775>. · [Zbl 1226.60113](#)
- [4] D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, *Bull. Amer. Math. Soc. (N.S.)*, 73 (1967), pp. 890–896, <https://doi.org/10.1090/S0002-9904-1967-11830-5>.
- [5] M. Bechtold, J. Vanderborght, O. Ippisch, and H. Vereecken, *Efficient random walk particle tracking algorithm for advective-dispersive transport in media with discontinuous dispersion coefficients and water contents*, *Water Res. Res.*, 47 (2011), W10526, <https://doi.org/10.1029/2010WR010267>.
- [6] S. Berezin and O. Zayats, *Skew Brownian Motion with dry friction: The Pugachev-Sveshnikov equation approach*, *Mater. Phys. Mech.*, 41 (2019), pp. 103–110.
- [7] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion – Facts and Formulae*, 2nd ed., *Probab. Appl.*, Birkhäuser, Basel, 2002.
- [8] M. Bossy, N. Champagnat, S. Maire, and D. Talay, *Probabilistic interpretation and random walk on spheres algorithms for the Poisson-Boltzmann equation in molecular dynamics*, *ESAIM Math. Model. Numer. Anal.*, 44 (2010), pp. 997–1048, <https://doi.org/10.1051/m2an/2010050>. · [Zbl 1204.82020](#)
- [9] R. Cantrell and C. Cosner, *Diffusion models for population dynamics incorporating individual behavior at boundaries: Applications to refuge design*, *Theor. Popul. Biol.*, 55 (1999), pp. 189–207, <https://doi.org/10.1006/tpbi.1998.1397>. · [Zbl 0958.92028](#)
- [10] Z.-Q. Chen and M. Zili, *One-dimensional heat equation with discontinuous conductance*, *Sci. China Math.*, 58 (2015), pp. 97–108, <https://doi.org/10.1007/s11425-014-4912-1>.
- [11] M. Decamps, A. De Schepper, and M. Goovaerts, *Applications of δ -function perturbation to the pricing of derivative securities*, *Phys. A*, 342 (2004), pp. 677–692, <https://doi.org/10.1016/j.physa.2004.05.035>.
- [12] F. Delay, P. Ackerer, and C. Danquigny, *Simulating solute transport in porous or fractured formations using random walks particle tracking: A review*, *Vadose Zone J.*, 4 (2005), pp. 360–379.
- [13] D. Dereudre, S. Mazzonetto, and S. Roelly, *An explicit representation of the transition densities of the skew Brownian motion with drift and two semipermeable barriers*, *Monte Carlo Methods Appl.*, 22 (2016), pp. 1–23, <https://doi.org/10.1515/mcma-2016-0100>. · [Zbl 1335.60151](#)
- [14] D. Dereudre, S. Mazzonetto, and S. Roelly, *Exact simulation of Brownian diffusions with drift admitting jumps*, *SIAM J. Sci. Comput.*, 39 (2017), pp. A711–A740, <https://doi.org/10.1137/16M107699X>. · [Zbl 1370.60113](#)
- [15] J. Eckhardt and G. Teschl, *Sturm-Liouville operators with measure-valued coefficients*, *J. Anal. Math.*, 120 (2013), pp. 151–224. · [Zbl 1315.34001](#)

- [16] P. Étoré, \textit{On random walk simulation of one-dimensional diffusion processes with discontinuous coefficients}, Electron. J. Probab., 11 (2006), pp. 249–275, <https://doi.org/10.1214/EJP.v11-311>. · Zbl 1112.60061
- [17] P. Étoré and A. Lejay, \textit{A Donsker theorem to simulate one-dimensional processes with measurable coefficients}, ESAIM Probab. Stat., 11 (2007), pp. 301–326, <https://doi.org/10.1051/ps:2007021>. · Zbl 1181.60123
- [18] P. Étoré and M. Martínez, \textit{Exact simulation of one-dimensional stochastic differential equations involving the local time at zero of the unknown process}, Monte Carlo Methods Appl., 19 (2013), pp. 41–71. · Zbl 1269.65007
- [19] W. Feller, \textit{Diffusion processes in one dimension}, Trans. Amer. Math. Soc., 77 (1954), pp. 1–31. · Zbl 0059.11601
- [20] W. Feller, \textit{The general diffusion operator and positivity preserving semi-groups in one dimension}, Ann. of Math. (2), 60 (1954), pp. 417–436. · Zbl 0057.09805
- [21] W. Feller, \textit{On second order differential operators}, Ann. of Math. (2), 61 (1955), pp. 90–105. · Zbl 0064.11301
- [22] W. Feller, \textit{Generalized second order differential operators and their lateral conditions}, Illinois J. Math., 1 (1957), pp. 459–504. · Zbl 0077.29102
- [23] W. Feller, \textit{On the intrinsic form for second order differential operator}, Illinois J. Math., 2 (1959), pp. 1–18.
- [24] E. R. Fernholz, T. Ichiba, and I. Karatzas, \textit{Two Brownian particles with rank-based characteristics and skew-elastic collisions}, Stochastic Process. Appl., 123 (2013), pp. 2999–3026. · Zbl 1296.60148
- [25] A. Gairat and V. Shcherbakov, \textit{Density of skew Brownian motion and its functionals with application in finance}, Math. Finance, 26 (2016), pp. 1069–1088, <https://doi.org/10.1111/mafi.12120>. · Zbl 1411.91555
- [26] E. M. Garon and J. V. Lambers, \textit{Modeling the diffusion of heat energy within composites of homogeneous materials using the uncertainty principle}, Comput. Appl. Math., 37 (2018), pp. 2566–2587, <https://doi.org/10.1007/s40314-017-0465-6>. · Zbl 1404.65194
- [27] B. Gaveau, M. Okada, and T. Okada, \textit{Second order differential operators and Dirichlet integrals with singular coefficients, I. Functional calculus of one-dimensional operators}, Tohoku Math. J. (2), 39 (1987), pp. 465–504. · Zbl 0653.35034
- [28] U. Gräwe, E. Deleersnijder, S. H. A. M. Shah, and A. W. Heemink, \textit{Why the Euler scheme in particle tracking is not enough: The shallow-sea pycnocline test case}, Ocean Dynam., 62 (2012), pp. 501–514, <https://doi.org/10.1007/s10236-012-0523-y>.
- [29] Y. Güldü, \textit{On discontinuous Dirac operator with eigenparameter dependent boundary and two transmission conditions}, Bound. Value Probl. (2016), 135, <https://doi.org/10.1186/s13661-016-0639-y>. · Zbl 1342.34027
- [30] H. Hoteit, R. Mose, A. Younes, F. Lehmann, and P. Ackerer, \textit{Three-dimensional modeling of mass transfer in porous media using the mixed hybrid finite elements and the random-walk methods}, Math. Geol., 34 (2002), pp. 435–456. · Zbl 1107.76401
- [31] K. Itô and H. P. McKean, \textit{Diffusion Processes and Their Sample Paths}, 2nd ed., Springer, Berlin, 1974.
- [32] E. M. LaBolle, G. E. Fogg, and A. F. B. Tompson, \textit{Random-walk simulation of transport in heterogeneous porous media: Local mass-conservation problem and implementation methods}, Water Res. Res., 32 (1996), pp. 583–593, <https://doi.org/10.1029/95WR03528>.
- [33] O. A. Ladyženskaja, V. J. Rivkind, and N. N. Ural'ceva, \textit{Equations aux dérivées partielles de type elliptique}, Monogr. Univ. Math. 31, Dunod, Paris, 1968.
- [34] O. A. Ladyženskaja, V. J. Rivkind, and N. N. Ural'ceva, \textit{Linear and Quasilinear Equations of Parabolic Type}, Transl. Math. Monogr. 33, American Mathematical Society, Providence, RI, 1968.
- [35] O. A. Ladyženskaja, V. J. Rivkind, and N. N. Ural'ceva, \textit{Classical solvability of diffraction problems for equations of elliptic and parabolic types}, Dokl. Akad. Nauk SSSR, 158 (1964), pp. 513–515.
- [36] J. Langebrake, L. Riotte-Lambert, C. W. Osenberg, and P. De Leenheer, \textit{Differential movement and movement bias models for marine protected areas}, J. Math. Biol., 64 (2012), pp. 667–696. · Zbl 1262.34049
- [37] H. Langer and W. Schenk, \textit{Knottung of one-dimensional Feller processes}, Math. Nachr., 113 (1983), pp. 151–161. · Zbl 0532.60066
- [38] J.-F. Le Gall, \textit{One-dimensional stochastic differential equations involving the local times of the unknown process}, in Stochastic Analysis, Lecture Notes in Math. 1095, Springer, Berlin, 1984, pp. 51–82.
- [39] A. Lejay, \textit{On the constructions of the skew Brownian motion}, Probab. Surv., 3 (2006), pp. 413–466. · Zbl 1189.60145
- [40] A. Lejay, \textit{Simulation of a stochastic process in a discontinuous layered medium}, Electron. Commun. Probab., 16 (2011), pp. 764–774. · Zbl 1243.60062
- [41] A. Lejay, L. Lenôtre, and G. Pichot, \textit{An exponential timestepping algorithm for diffusion with discontinuous coefficients}, J. Comput. Phys., 396 (2019), pp. 888–904.
- [42] A. Lejay and M. Martínez, \textit{A scheme for simulating one-dimensional diffusion processes with discontinuous coefficients}, Ann. Appl. Probab., 16 (2006), pp. 107–139. · Zbl 1094.60056
- [43] A. Lejay and G. Pichot, \textit{Simulating diffusion processes in discontinuous media: A numerical scheme with constant time steps}, J. Comput. Phys., 231 (2012), pp. 7299–7314. · Zbl 1284.65007
- [44] A. Lejay and G. Pichot, \textit{Simulating diffusion processes in discontinuous media: Benchmark tests}, J. Comput. Phys., 314 (2016), pp. 384–413, <https://doi.org/10.1016/j.jcp.2016.03.003>. · Zbl 1349.65019
- [45] F. T. Lindstrom and F. Oberhettinger, \textit{A note on a Laplace transform pair associated with mass transport in porous media and heat transport problems}, SIAM J. Appl. Math., 29 (1975), pp. 288–292. · Zbl 0325.44001
- [46] A. Lipton, \textit{Oscillating Bachelier and Black–Scholes formulas}, World Scientific, in Financial Engineering, Hackensack, NJ, 2018, pp. 371–394, <https://doi.org/10.1142/10425>.

- [47] M. Martinez, \textit{Interprétations probabilistes d'opérateurs sous forme divergence et analyse de méthodes numériques probabilistes associées}, Ph.D. thesis, Université de Provence, Marseille, France, 2004.
- [48] H. P. McKean, Jr., \textit{Elementary solutions for certain parabolic partial differential equations}, Trans. Amer. Math. Soc., 82 (1956), pp. 519–548, <https://doi.org/10.1090/S0002-9947-1956-0087012-3>.
- [49] T. Okada, \textit{Asymptotic behavior of skew conditional heat kernels on graph networks}, Canad. J. Math., 45 (1993), pp. 863–878, <https://doi.org/10.4153/CJM-1993-049-6>. · Zbl 0799.58084
- [50] J. Pitman and M. Yor, \textit{Hitting, occupation and inverse local times of one-dimensional diffusions: Martingale and excursion approaches}, Bernoulli, 9 (2003), pp. 1–24, <https://doi.org/10.3150/bj/1068129008>. · Zbl 1024.60032
- [51] P. Puri and P. Kythe, \textit{Some inverse Laplace transforms of exponential form}, Z. Angew. Math. Phys., 39 (1988), pp. 150–156, <https://doi.org/10.1007/BF00945761>. · Zbl 0644.44002
- [52] J. M. Ramirez, E. A. Thomann, E. C. Waymire, J. Chastanet, and B. D. Wood, \textit{A note on the theoretical foundations of particle tracking methods in heterogeneous porous media}, Water Resour. Res., 44 (2008), W01501, <https://doi.org/10.1029/2007WR005914>.
- [53] P. Salamon, D. Fernández-García, and J. J. Gómez-Hernández, \textit{A review and numerical assessment of the random walk particle tracking method}, J. Contam. Hydrol., 87 (2006), pp. 277–305, <https://doi.org/10.1016/j.jconhyd.2006.05.005>.
- [54] D. Spivakovskaya, A. W. Heemink, and E. Deleersnijder, \textit{Lagrangian modelling of multi-dimensional advection-diffusion with space-varying diffusivities: Theory and idealized test cases}, Ocean Dynam., 57 (2007), pp. 189–203.
- [55] D. W. Stroock, \textit{Diffusion semigroups corresponding to uniformly elliptic divergence form operators}, in Séminaire de Probabilités, XXII, Lecture Notes in Math. 1321, Springer, Berlin, 1988, pp. 316–347, <https://doi.org/10.1007/BFb0084145>.
- [56] D. Thomson, W. Physick, and R. Maryon, \textit{Treatment of interfaces in random walk dispersion models}, J. Appl. Meteorol., 36 (1997), pp. 1284–1295.
- [57] G. Uffink, \textit{A random walk method for the simulation of macrodispersion in a stratified aquifer}, in Relation of Groundwater Quantity and Quality, IAHS Publication 146, International Association of Hydrological Sciences, Wallingford, United Kingdom, 1985, pp. 103–114.
- [58] J. B. Walsh, \textit{A diffusion with discontinuous local time}, in Temps locaux, Vol. 52-53, Société Mathématique de France, Paris, 1978, pp. 37–45.
- [59] S. Wang, S. Song, and Y. Wang, \textit{Skew Ornstein–Uhlenbeck processes and their financial applications}, J. Comput. Appl. Math., 273 (2015), pp. 363–382, <https://doi.org/10.1016/j.cam.2014.06.023>. · Zbl 1304.60046
- [60] E. Zauderer, \textit{Partial Differential Equations of Applied Mathematics}, 3rd ed., Pure Appl. Math., Wiley, Hoboken, NJ, 2006, <https://doi.org/10.1002/9781118033302>.
- [61] M. Zhang, \textit{Calculation of diffusive shock acceleration of charged particles by skew Brownian motion}, Astrophys. J., 541 (2000), pp. 428–435.
- [62] C. Zheng and G. D. Bennett, \textit{Applied Contaminant Transport Modelling}, 2nd ed., Wiley-Interscience, New York, 2002.

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