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A conservative discretization of the shallow-water equations on triangular grids. (English)

Zbl 1416.86003
J. Comput. Phys. 375, 871-900 (2018).

Summary: A structure-preserving discretization of the shallow-water equations on unstructured spherical grids is introduced. The unstructured grids that we consider have triangular cells with a C-type staggering of variables, where scalar variables are located at centres of grid cells and normal components of velocity are placed at cell boundaries. The staggering necessitates reconstructions and these reconstructions are build into the algorithm such that the resulting discrete equations obey a weighted weak form. This approach, combined with a mimetic discretization of the differential operators of the shallow-water equations, provides a conservative discretization that preserves important aspects of the mathematical structure of the continuous equations, most notably the simultaneous conservation of quadratic invariants such as energy and enstrophy. The structure-preserving nature of our discretization is confirmed through theoretical analysis and through numerical experiments on two different triangular grids, a symmetrized icosahedral grid of nearly uniform resolution and a non-uniform triangular grid whose resolution increases towards the poles.

MSC:

86A05	Hydrology, hydrography, oceanography
86A10	Meteorology and atmospheric physics
86-08	Computational methods for problems pertaining to geophysics
65M50	Mesh generation, refinement, and adaptive methods for the numerical solution of initial value and initial-boundary value problems involving PDEs

Keywords:

geophysical fluid dynamics; shallow water equations; unstructured grids; structure-preserving discretization; conservation laws; weak form

Software:

chammp; ICON; MPAS-Ocean

Full Text: DOI

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