Bridy, Andrew; **Tucker, Thomas J.**

Finite index theorems for iterated Galois groups of cubic polynomials. (English) Zbl 07051737

Math. Ann. 373, No. 1-2, 37-72 (2019).

Summary: Let K be a number field or a function field. Let $f \in K(x)$ be a rational function of degree $d \geq 2$, and let $\beta \in \mathbb{P}^1(\overline{K})$. For all $n \in \mathbb{N} \cup \{\infty\}$, the Galois groups $G_n(\beta) = \mathrm{Gal}(K(f^{-n}(\beta))/K(\beta))$ embed into $\mathrm{Aut}(T_n)$, the automorphism group of the d-ary rooted tree of level n. A major problem in arithmetic dynamics is the arboreal finite index problem: determining when $[\mathrm{Aut}(T_\infty) : G_\infty(\beta)] < \infty$. When f is a cubic polynomial and K is a function field of transcendence degree 1 over an algebraic extension of $\mathbb{Q}$, we resolve this problem by proving a list of necessary and sufficient conditions for finite index. This is the first result that gives necessary and sufficient conditions for finite index, and can be seen as a dynamical analog of the Serre Open Image Theorem. When K is a number field, our proof is conditional on both the *abc* conjecture for K and Vojta's conjecture for blowups of $\mathbb{P}^1 \times \mathbb{P}^1$. We also use our approach to solve some natural variants of the finite index problem for modified trees.

MSC:

37P15 Dynamical systems over global ground fields
11G50 Heights
11R32 Galois theory
14G25 Global ground fields in algebraic geometry
37P05 Arithmetic and non-Archimedean dynamical systems involving polynomial and rational maps
37P30 Height functions; Green functions; invariant measures in arithmetic and non-Archimedean dynamical systems

Full Text: DOI arXiv

References:

[1] Bugeaud, Y.; Corvaja, P.; Zannier, U., An upper bound for the G.C.D. of \(a^n-1\) and \(b^n-1\), Math. Z., 243, 79-84, (2003) · Zbl 1021.11001

[2] Benedetto, R.L., Faber, X., Hutz, B., Juul, J., Yasufuku, Y.: A large arboreal Galois representation for a cubic postcritically finite polynomial. arXiv:1612.03358 (2016) · Zbl 1427.11118

[3] Bombieri, E., Gubler, W.: Heights in Diophantine Geometry, New Mathematical Monographs, vol. 4. Cambridge University Press, Cambridge (2006) · Zbl 1115.11034

[4] Bridy, A., Ghioca, D., Hsia, L.-C., Tucker, T.J.: Iterated galois groups of quadratic polynomials. In preparation (2017)

[5] Benedetto, RL; Ghioca, D.; Kurlberg, P.; Tucker, TJ, A case of the dynamical Mordell-Lang conjecture, Math. Ann., 352, 1-26, (2012) · Zbl 1285.37021

[6] Bell, J.P., Ghioca, D., Tucker, T.J.: The Dynamical Mordell-Lang Conjecture, Mathematical Surveys and Monographs, vol. 210. American Mathematical Society, Providence (2016) · Zbl 1362.11001

[7] Bridy, A., Ingram, P., Jones, R., Juul, R., Levy, A., Manes, M., Rubinstein-Salzedo, S., Silverman, J.H.: Finite ramification for preimage fields of postcritically finite morphisms. arXiv:1511.00194 (2017). To appear in Math. Res. Lett · Zbl 1391.14004

[8] Boston, N.; Jones, R., Arboreal Galois representations, Geom. Dedicata, 124, 27-35, (2007) · Zbl 1206.11069

[9] Boston, N.; Jones, R., The image of an arboreal Galois representation, Pure Appl. Math. Q., 5, 213-225, (2009) · Zbl 1167.11011

[10] Bridy, A.; Tucker, TJ, \(ABC\) implies a Zsigmondy principle for ramification, J. Number Theory, 182, 296-310, (2018) · Zbl 1407.37128

[11] Cremona, J.E.: On the Galois groups of the iterates of \(x^2+1\). Mathematika \textbf{36}(2):259-261 (1989-1990) · Zbl 0699.12018

[12] Call, GS; Silverman, JH, Canonical heights on varieties with morphisms, Compositio Math., 89, 163-205, (1993) · Zbl 0826.14015

[13] Corvaja, P.; Zannier, U., A lower bound for the height of a rational function at \(S\)-unit points, Monatsh. Math., 144, 203-224, (2005) · Zbl 1086.11035

[14] Doyle, J., Levy, A., Tucker, T.J.: Quadratic polynomials over function fields are eventually stable. In preparation (2017)

[15] Favre, C., Gauthier, T.: Classification of special curves in the space of cubic polynomials. arXiv:1603.05126 (2016). To appear in Internat. Math. Res. Notices · Zbl 1415.14013

[16] Fein, B.; Schacher, M., Properties of iterates and composites of polynomials, J. Lond. Math. Soc. (2), 54, 489-497, (1996) · Zbl 0865.12003

[17] Ghioca, D.; Krieger, H.; Nguyen, KD; Ye, H., The dynamical André-Oort conjecture: unicritical polynomials, Duke Math. J., 166, 1-25, (2017) · Zbl 1417.37172

[18] Gratton, C.; Nguyen, K.; Tucker, TJ, \(ABC\) implies primitive prime divisors in arithmetic dynamics, Bull. Lond. Math. Soc., 45, 1194-1208, (2013) · Zbl 1291.37121

[19] Ghioca, D.; Nguyen, K.; Tucker, TJ, Portraits of preperiodic points for rational maps, Math. Proc. Camb. Philos. Soc., 159, 165-186, (2015) · Zbl 1371.11109

[20] Granville, A., \(ABC\) allows us to count squarefrees, Int. Math. Res. Not., 1998, 991-1009, (1998) · Zbl 0924.11018

[21] Grell, G.: Profinite iterated monodromy groups for cubic polynomials. In preparation (2017)

[22] Ghioca, D.; Tucker, TJ, Periodic points, linearizing maps, and the dynamical Mordell-Lang problem, J. Number Theory, 129, 1392-1403, (2009) · Zbl 1186.14047

[23] Ghioca, D.; Tucker, TJ; Zieve, ME, Intersections of polynomials orbits, and a dynamical Mordell-Lang conjecture, Invent. Math., 171, 463-483, (2008) · Zbl 1191.14027

[24] Gubler, W., Equidistribution over function fields, Manuscripta Math., 127, 485-510, (2008) · Zbl 1189.14030

[25] Ghioca, D., Ye, H.: The dynamical Andre-Oort conjecture for cubic polynomials. arXiv:1603.05303 (2016). To appear in Int. Math. Res. Not

[26] Hindes, W., Points on elliptic curves parametrizing dynamical Galois groups, Acta Arith., 159, 149-167, (2013) · Zbl 1296.14017

[27] Hindes, W., The arithmetic of curves defined by iteration, Acta Arith., 169, 1-27, (2015) · Zbl 1330.14032

[28] Hamblen, S.; Jones, R.; Madhu, K., The density of primes in orbits of \(z^d+c\), Int. Math. Res. Not. IMRN, 2015, 1924-1958, (2015) · Zbl 1395.11128

[29] Hindry, M., Silverman, J.H.: Diophantine Geometry, Graduate Texts in Mathematics, vol. 201. Springer, New York (2000). (An introduction) · Zbl 0948.11023

[30] Huang, K.: Generalized greatest common divisors for the orbits under rational functions. arXiv:1702.03881 (2017)

[31] Ingram, P., Arboreal Galois representations and uniformization of polynomial dynamics, Bull. Lond. Math. Soc., 45, 301-308, (2013) · Zbl 1280.37058

[32] Ingram, P.; Silverman, JH, Primitive divisors in arithmetic dynamics, Math. Proc. Camb. Philos. Soc., 146, 289-302, (2009) · Zbl 1242.11012

[33] Juul, J.; Kurlberg, P.; Madhu, K.; Tucker, TJ, Wreath products and proportions of periodic points, Int. Math. Res. Not. IMRN, 2016, 3944-3969, (2016) · Zbl 1404.37124

[34] Jones, Rafe; Levy, Alon, Eventually stable rational functions, Int. J. Number Theory, 13, 2299-2318, (2017) · Zbl 1391.37072

[35] Jones, R.; Manes, M., Galois theory of quadratic rational functions, Comment. Math. Helv., 89, 173-213, (2014) · Zbl 1316.11104

[36] Jones, R., Iterated Galois towers, their associated martingales, and the \(p\)-adic Mandelbrot set, Compos. Math., 143, 1108-1126, (2007) · Zbl 1166.11040

[37] Jones, R., The density of prime divisors in the arithmetic dynamics of quadratic polynomials, J. Lond. Math. Soc. (2), 78, 523-544, (2008) · Zbl 1193.37144

[38] Jones, R.: Galois representations from pre-image trees: an arboreal survey. In: Actes de la Conférence "Théorie des Nombres et Applications", Publ. Math. Besançon Algèbre Théorie Nr., vol. 2013, pp. 107-136. Presses Univ. Franche-Comté, Besançon (2013) · Zbl 1307.11069

[39] Jones, R., Fixed-point-free elements of iterated monodromy groups, Trans. Am. Math. Soc., 367, 2023-2049, (2015) · Zbl 1385.37063

[40] Juul, J.: Iterates of generic polynomials and generic rational functions. arXiv:1410.3814 (2016). To appear in Trans. Am. Math. Soc

[41] Krieger, H., Primitive prime divisors in the critical orbit of \(z^d+c\), Int. Math. Res. Not. IMRN, 2013, 5498-5525, (2013) · Zbl 1329.11020

[42] Looper, N.: Dynamical Galois groups of trinomials and Odoni's conjecture. arXiv:1609.03398v2 (2016)

[43] Morton, P.; Silverman, JH, Rational periodic points of rational functions, Int. Math. Res. Not., 1994, 97-110, (1994) · Zbl 0819.11045

[44] Medvedev, A.; Scanlon, T., Invariant varieties for polynomial dynamical systems, Ann. Math. (2), 179, 81-177, (2014) · Zbl 1347.37145

[45] Nguyen, K., Some arithmetic dynamics of diagonally split polynomial maps, Int. Math. Res. Not. IMRN, 2015, 1159-1199, (2015) · Zbl 1391.37073

[46] Odoni, RWK, The Galois theory of iterates and composites of polynomials, Proc. Lond. Math. Soc. (3), 51, 385-414, (1985) · Zbl 0622.12011

[47] Odoni, RWK, Realising wreath products of cyclic groups as Galois groups, Mathematika, 35, 101-113, (1988) · Zbl 0662.12010

[48] Pink, R.: Finiteness and liftability of postcritically finite morphisms in arbitrary characteristic. arXiv:1305.2841 (2013)

[49] Pink, R.: Profinite iterated monodromy groups arising from quadratic morphisms with infinite postcritical orbits. arXiv:1309.5804

(2013)

[50] Pink, R.: Profinite iterated monodromy groups arising from quadratic polynomials. arXiv:1307.5678 (2013)

[51] Reznick, B., When is the iterate of a formal power series odd?, J. Aust. Math. Soc. Ser. A, 28, 62-66, (1979) · Zbl 0441.30032

[52] Ritt, JF, On the iteration of rational functions, Trans. Am. Math. Soc., 21, 348-356, (1920) · Zbl 47.0312.01

[53] Ritt, JF, Permutable rational functions, Trans. Am. Math. Soc., 25, 399-448, (1923) · Zbl 49.0712.02

[54] Serre, J-P, Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, Invent. Math., 15, 259-331, (1972) · Zbl 0235.14012

[55] Silverman, JH, Generalized greatest common divisors, divisibility sequences, and Vojta's conjecture for blowups, Monatsh. Math., 145, 333-350, (2005) · Zbl 1197.11070

[56] Silverman, J.H.: The Arithmetic of Dynamical Systems, Graduate Texts in Mathematics, vol. 241. Springer, New York (2007)

[57] Stoll, M., Galois groups over \({ Q}\) of some iterated polynomials, Arch. Math. (Basel), 59, 239-244, (1992) · Zbl 0758.11045

[58] Vojta, P.: Diophantine Approximations and Value Distribution Theory, Lecture Notes in Mathematics, vol. 1239. Springer, Berlin (1987) · Zbl 0609.14011

[59] Xie, J.: The Dynamical Mordell-Lang Conjecture for polynomial endomorphisms of the affine plane. arXiv:1503.00773 (2015). To appear in Astérisque

[60] Yamanoi, K., The second main theorem for small functions and related problems, Acta Math., 192, 225-294, (2004) · Zbl 1203.30035

[61] Yuan, X., Big line bundles over arithmetic varieties, Invent. Math., 173, 603-649, (2008) · Zbl 1146.14016