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Sharp phase transition for the random-cluster and Potts models via decision trees. (English)

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Summary: We prove an inequality on decision trees on monotonic measures which generalizes the OSSS inequality on product spaces. As an application, we use this inequality to prove a number of new results on lattice spin models and their random-cluster representations. More precisely, we prove that

- For the Potts model on transitive graphs, correlations decay exponentially fast for $\beta < \beta_c$.
- For the random-cluster model with cluster weight $q \geq 1$ on transitive graphs, correlations decay exponentially fast in the subcritical regime and the cluster-density satisfies the mean-field lower bound in the supercritical regime.
- For the random-cluster models with cluster weight $q \geq 1$ on planar quasi-transitive graphs $\mathbb{G}$,

$$\frac{p_c(\mathbb{G})p_c(\mathbb{G}^*)}{(1-p_c(\mathbb{G}))(1-p_c(\mathbb{G}^*))} = q.$$

  As a special case, we obtain the value of the critical point for the square, triangular and hexagonal lattices. (This provides a short proof of a result of Beffara and the first author dating from 2012.)

These results have many applications for the understanding of the subcritical (respectively disordered) phase of all these models. The techniques developed in this paper have potential to be extended to a wide class of models including the Ashkin-Teller model, continuum percolation models such as Voronoi percolation and Boolean percolation, super-level sets of massive Gaussian free field, and the random-cluster and Potts models with infinite range interactions.

MSC:

60K35 Interacting random processes; statistical mechanics type models; percolation theory

Keywords:

percolation model; Potts model; randomized algorithm; sharp threshold; exponential decay

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