Zagier, Don B.
The arithmetic and topology of differential equations. (English) [Zbl 06979410]

In this survey paper, the author presents connections between the theory of differential equations and several other areas of mathematics, ranging from number theory and algebraic geometric to topology. Apéry’s proof of the irrationality of $\zeta(3)$ serves as a starting point and motivating example. The Apéry numbers, defined by satisfying a certain recursion, are related to the Picard-Fuchs equation of a family of K3 surfaces.

Many different topics are covered and some basic theory is introduced for that. Differential forms and de Rham’s theorem are reviewed to explain cohomology and Picard-Fuchs equations. Modular forms are introduced and relations to differential equations are explicated. Étale cohomology is introduced in the context of the Weil conjectures which relate point counting over finite fields to topology. The different cohomology theories lead to periods and the idea of motives, which is illustrated through several examples.

While a lot of expected properties of motives are still conjectural, in Section 7 various examples are given for which the motivic theory suggests concrete statements that can be checked using other means. This is the only section that is not entirely expository.

In the final sections D. Zagier gives a brief introduction to mirror symmetry and Gromov-Witten invariants, in particular for Fano manifolds. He explains the Gamma conjecture for Fano manifolds which relates the asymptotic behaviour of the quantum differential equation of a Fano manifold to a certain cohomology class of the manifold, called the Gamma class. The paper closes with a section on open questions and further directions.

Everything is illustrated through plentiful examples and in particular the Apéry numbers are revisited often in different contexts.

For the entire collection see [Zbl 1396.00017].

Reviewer: Konstantin Jakob (Cambridge)

MSC:

11F67 Special values of automorphic $L$-series, periods of automorphic forms, cohomology, modular symbols

14N35 Gromov-Witten invariants, quantum cohomology, Gopakumar-Vafa invariants, Donaldson-Thomas invariants (algebro-geometric aspects)

12H20 Abstract differential equations

34M15 Algebraic aspects (differential-algebraic, hypertranscendence, group-theoretical) of ordinary differential equations in the complex domain

14J32 Calabi-Yau manifolds (algebro-geometric aspects)

14J33 Mirror symmetry (algebro-geometric aspects)

11F11 Holomorphic modular forms of integral weight

11-02 Research exposition (monographs, survey articles) pertaining to number theory

Keywords:

Picard-Fuchs differential equations; Gamma conjecture; $L$-series

Full Text: DOI

References:


[29] Sergey Galkin and Alexander Usnich, Mutation of polynomials. IPMU 10-0100.


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