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Hesse pencils and 3-torsion structures. (English) Zbl 1403.14030

SIGMA, Symmetry Integrability Geom. Methods Appl. 14, Paper 102, 13 p. (2018).

Let E be an elliptic curve defined over a perfect field k of characteristic different from 3 and let $E[3]$ be its 3-torsion subgroup. Let $F_E \in k[X,Y,Z]$ be a homogenous equation for E and consider the Hesse pencil of E as the zero set $\mathcal{E}_E$ of $tF_E + \mathrm{Hess}(F_E)$ over $k(t)$ (where $\mathrm{Hess}(F_E)$ is the Hessian of F_E). For any $t_0 \in \mathbb{P}^1(k)$, let $\mathcal{E}_E^{t_0}$ be the reduction of $\mathcal{E}_E$ at t_0: by looking at flex points, the authors show an explicit correspondence between $E[3]$ and $\mathcal{E}_E^{t_0}[3]$ (and the associated Weil pairing) whenever $\mathcal{E}_E^{t_0}$ is nonsingular (and $\mathrm{char}(k) \neq 2$). This is used to (re)prove (essentially via linear algebra) that two curves E and E' have simplectically equivalent Galois representations on their 3-torsion (i.e. $E[3] \simeq E'[3]$ via a $\mathrm{Gal}(\overline{k}/k)$-isomorphism compatible with the Weil pairing) if and only if $E' \simeq \mathcal{E}_E^{t_0}$ for some $t_0 \in \mathbb{P}^1(k)$. The proof has to be modified if $\mathrm{char}(k) = 2$: indeed $\mathrm{Hess}(F_E) \equiv 0$ in that case and, to overcome this difficulty, the authors present explicit equations for the Hesse pencils which still have the property of having the same flex points of the initial curve. Since this was the main ingredient of the proof in the cases $\mathrm{char}(k) \neq 2, 3$, they can start again a linear algebra machinery providing a complete proof of the above statement for all perfect fields of characteristic different from 3.

Reviewer: Andrea Bandini (Parma)

MSC:

14D10 Arithmetic ground fields (finite, local, global) and families or fibrations
14G99 Arithmetic problems in algebraic geometry; Diophantine geometry

Keywords:

Hesse pencil; Galois representation; torsion points; elliptic curves

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