Kılıçer, Pınar; **Labrande, Hugo**; **Lercier, Reynald**; **Ritzenthaler, Christophe**; **Sijsling, Jeroen**; **Streng, Marco**

Plane quartics over $\mathbb{Q}$ with complex multiplication. (English) Zbl 1409.14051

Acta Arith. 185, No. 2, 127-156 (2018).

The paper deals with Jacobians of genus 3 curves and sextic CM fields. The authors show that there are exactly 37 isomorphism classes of CM fields K for which there exist principally polarized abelian threefolds $A/\mathbb{Q}$ with field of moduli $\mathbb{Q}$ and $\mathrm{End}(A) \cong \mathcal{O}_K$. In the proof of this result they list 37 cyclic sextic CM fields whose maximal orders give rise to CM curves of genus 3 with field of moduli $\mathbb{Q}$. Some of these curves have been computed before this work, namely hyperelliptic and Picard curves. Thus, this paper completes the list of curves of genus 3 over $\mathbb{Q}$ whose endomorphism rings over $\overline{\mathbb{Q}}$ are maximal orders of sextic fields. The authors consider the case of plane quartics with trivial automorphism group. The construction of these curves follows the classical path. They determine first the period matrices, and then they compute corresponding invariants. The curves are reconstructed from rational approximations of these invariants. The resulting equations are given at the end of the paper.

The authors point out that new phenomena might occur for plane quartics. These phenomena do not have an exact equivalent in lower genus, so they would require new theoretical development in order to be fully explained.

Reviewer: Tony Ezome (Libreville)

MSC:

14H25 Arithmetic ground fields for curves
11G15 Complex multiplication and moduli of abelian varieties
11Y40 Algebraic number theory computations
14H45 Special algebraic curves and curves of low genus
14K22 Complex multiplication and abelian varieties
14K25 Theta functions and abelian varieties
14Q05 Computational aspects of algebraic curves
13A50 Actions of groups on commutative rings; invariant theory

Keywords:

complex multiplication; genus 3; plane quartics; explicit aspects

Software:

endomorphisms; genus3; GitHub; Magma; PARI/GP; quartic_reconstruction; RECIP; SageMath

Full Text: DOI

References:

[1] J. S. Balakrishnan, S. Ionica, K. Lauter, and C. Vincent, Genus 3, package available at https://github.com/christellevincent/genus3. [3] J. S. Balakrishnan, S. Ionica, K. Lauter, and C. Vincent, Constructing genus-3 hyperelliptic Jacobians with CM, LMS J. Comput. Math. 19 (2016), suppl. A, 283-300. · Zbl 1404.11085

[2] C. Batut, K. Belabas, D. Benardi, H. Cohen, and M. Olivier, User's Guide to PARI/GP, 1998, https://pari.math.u-bordeaux.fr. [5] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), 235-265. [6] I. Bouw, J. Cooley, K. Lauter, E. Lorenzo Garc ́ıa, M. Manes, R. Newton, and E. Ozman, Bad reduction of genus three curves with complex multiplication, Women in Numbers Europe, Assoc. Women Math. Ser. 2, Springer, Cham, 2015, 109-151. [7] F. Bouyer and M. Streng, Examples of CM curves of genus two defined over the reflex field, LMS J. Comput. Math. 18 (2015) 507-538. [8] R. Br ̈oker and P. Stevenhagen, Elliptic curves with a given number of points, in: Algorithmic Number Theory, Lecture Notes in Comput. Sci. 3076, Springer, Berlin, 2004, 117-131.

[3] E. Costa, N. Mascot, J. Sijsling, and J. Voight, Rigorous computation of the endomorphism ring of a Jacobian, arXiv:1705.09248v3 (2018). · Zbl 07009723

[4] B. Deconinck, M. Heil, A. Bobenko, M. van Hoeij, and M. Schmies, Computing Riemann theta functions, Math. Comp. 73

(2004), 1417-1442. [11] B. Deconinck and M. van Hoeij, Computing Riemann matrices of algebraic curves, Phys. D, 152/153 (2001), 28-46. [12] J. Dixmier, On the projective invariants of quartic plane curves, Adv. Math. 64 (1987), 279-304. · Zbl 1092.33018

[5] R. Dupont, Moyenne arithm´etico-g´eom´etrique, suites de Borchardt et applications, PhD thesis, ´Ecole polytechnique, Palaiseau, 2006.

[6] A.-S. Elsenhans, Good models for cubic surfaces, https://math.uni-paderborn.de/ fileadmin/mathematik/AG-Computeralgebra/Preprints-elsenhans/red 5.pdf. [15] A.-S. Elsenhans, Explicit computations of invariants of plane quartic curves, J. Symbolic Comput. 68 (2015), part 2, 109-115.

[7] A. Fiorentino, Weber's formula for the bitangents of a smooth plane quartic, arXiv:1612.02049 (2016). [17] M. Girard and D. R. Kohel, Classification of genus 3 curves in special strata of the moduli space, in: F. Hess et al. (eds.), Algorithmic Number Theory, Lecture Notes in Comput. Sci. 4076, Springer, Berlin, 2006, 346-360. [18] E. Z. Goren and K. E. Lauter, Class invariants for quartic CM fields, Ann. Inst. Fourier (Grenoble) 57 (2007), 457-480.

[8] B. H. Gross and J. Harris, On some geometric constructions related to theta characteristics, in: Contributions to Automorphic Forms, Geometry, and Number Theory, Johns Hopkins Univ. Press, Baltimore, MD, 2004, 279-311. [20] J. Gu'ardia, On the Torelli problem and Jacobian Nullwerte in genus three, Michigan Math. J. 60 (2011), 51-65. [21] J.-I. Igusa, Theta Functions, Grundlehren Math. Wiss. 194, Springer, New York, 1972.

[9] P. Kılı¸cer, The CM class number one problem for curves, PhD thesis, Univ. Leiden and Univ. de Bordeaux, 2016.

[10] P. Kılı¸cer, H. Labrande, R. Lercier, C. Ritzenthaler, J. Sijsling, and M. Streng, CM plane quartics, available at https://arxiv.org/src/1701.06489/an · Zbl 1409.14051

[11] P. Kılı¸cer, K. Lauter, E. Lorenzo Garc´ıa, R. Newton, E. Ozman, and M. Streng, A bound on the primes of bad reduction for CM curves of genus 3, arXiv:1609.05826v3 (2018).

[12] P. Kılı¸cer, E. Lorenzo Garc´ıa, and M. Streng, Primes dividing invariants of CM Picard curves, arXiv:1801.04682 (2018).

[13] P. Kılı¸cer and M. Streng, Rational CM points and class polynomials for genus three, in preparation, 2016.

[14] K. Koike and A. Weng, Construction of CM Picard curves, Math. Comp. 74 (2005), 499-518. [28] S. Koizumi, The fields of moduli for polarized abelian varieties and for curves, Nagoya Math. J. 48 (1972), 37-55.

[15] H. Labrande, Calculating theta functions in genus 3, package available at http:// hlabrande.fr/pubs/fastthetaconstantsgenus3.m. · Zbl 1430.11167

[16] H. Labrande, Explicit computation of the Abel-Jacobi map and its inverse, PhD thesis, Univ. de Lorraine and Univ. of Calgary, 2016. [31] H. Labrande and E. Thom´e, Computing theta functions in quasi-linear time in genus two and above, LMS J. Comput. Math. 19 (2016), suppl. A, 163-177. [32] S. Lang, Complex Multiplication, Grundlehren Math. Wiss. 255, Springer, New York, 1983. · Zbl 1361.14028

[17] J.-C. Lario and A. Somoza, A note on Picard curves of CM-type, arXiv:1611.02582 (2016).

[18] K. Lauter, Geometric methods for improving the upper bounds on the number of rational points on algebraic curves over finite fields, J. Algebraic Geom. 10 (2001), 19-36. [35] K. Lauter and B. Viray, An arithmetic intersection formula for denominators of Igusa class polynomials, Amer. J. Math. 137 (2015), 497-533.

[19] R. Lercier and C. Ritzenthaler, Invariants and reconstructions for genus 2 curves in any characteristic, 2008; available in Magma 2.15 and later. [37] R. Lercier and C. Ritzenthaler, Hyperelliptic curves and their invariants: geometric, arithmetic and algorithmic aspects, J. Algebra 372 (2012), 595-636. [38] R. Lercier, C. Ritzenthaler, F. Rovetta, and J. Sijsling, Parametrizing the moduli space of curves and applications to smooth plane quartics over finite fields, LMS J. Comput. Math. 17 (2014), suppl. A, 128-147. [39] R. Lercier, C. Ritzenthaler, and J. Sijsling, Fast computation of isomorphisms of hyperelliptic curves and explicit descent, in: E. W. Howe and K. S. Kedlaya (eds.), ANTS X—Proc. Tenth Algorithmic Number Theory Symposium, Math. Sci. Publ., Berkeley, CA, 2013, 463-486. · Zbl 1333.14060

[20] R. Lercier, C. Ritzenthaler, and J. Sijsling, Explicit Galois obstruction and descent for hyperelliptic curves with tamely cyclic reduced automorphism group, Math. Comp. 85 (2016), 2011-2045. · Zbl 1343.14047

[21] R. Lercier, C. Ritzenthaler, and J. Sijsling, quartic reconstruction, a Magma package for reconstructing plane quartics from Dixmier-Ohno invariants, https://github.com/ JRSijsling/quartic reconstruction, 2016.

[22] R. Lercier, C. Ritzenthaler, and J. Sijsling, Reconstructing plane quartics from their invariants, arXiv:1606.05594 (2016).

[23] E. Lorenzo Garc´ıa, R. Lercier, and C. Ritzenthaler, Reduction type of non-hyperelliptic genus 3 curves, in preparation, 2017. [44] J.-F. Mestre, Construction de courbes de genre 2 'a partir de leurs modules, in: Effective Methods in Algebraic Geometry, Progr. Math. 94, Birk¨auser, Boston, 1991, 313-334.

[24] H. E. Rauch and H. M. Farkas, Theta Functions with Applications to Riemann Surfaces, Williams \& Wilkins, Baltimore, MD, 1974. · Zbl 0292.30015

[25] S. M. van Rijnswou, Testing the equivalence of planar curves, PhD thesis, Technische Univ. Eindhoven, 2001. [47] C. Ritzenthaler, Point counting on genus 3 non hyperelliptic curves, in: Algorithmic Number Theory, Lecture Notes in Comput. Sci. 3076, Springer, Berlin, 2004, 379-394. · Zbl 0982.13004

[26] The Sage Developers, SageMath, the Sage Mathematics Software System (Version 7.4), 2016, http://www.sagemath.org. [49] J.-P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. (2) 88 (1968), 492-517. [50] G. Shimura, On the field of rationality for an abelian variety, Nagoya Math. J. 45 (1972), 167-178. [51] G. Shimura, On abelian varieties with complex multiplication, Proc. London Math. Soc. (3) 34 (1977), 65-86.

[27] G. Shimura and Y. Taniyama, Complex multiplication of Abelian varieties and its applications to number theory, Publ. Math. Soc. Japan 6, Math. Soc. Japan, Tokyo, 1961. · Zbl 0112.03502

[28] D. Simon, Solving quadratic equations using reduced unimodular quadratic forms, preprint at http://www.math.unicaen.fr/⊠simon/maths/qfsolve. 2013. · Zbl 1078.11072

[29] A.-M. Spallek, Kurven vom Geschlecht 2 und ihre Anwendung in Publik-Key-Kryptosystemen, PhD thesis, Inst. f¨ur Experimentelle Math., Essen, 1994. [55] M. Stoll, Reduction theory of point clusters in projective space, Groups Geom. Dyn. 5 (2011), 553-565.

[30] M. Streng, RECIP - REpository of Complex multiPlication SageMath code (formerly Sage package for using Shimura's reciprocity law for Siegel modular functions), http://www.math.leidenuniv.nl/⊠streng/recip.

[31] M. Streng, Computing Igusa class polynomials, Math. Comp. 83 (2014), 275-309. · Zbl 1322.11066

[32] J. Tate, Classes d'isog´enie des vari´et´es ab´eliennes sur un corps fini (d'apr'es Honda), in: S´eminaire Bourbaki 1968/69, Lecture Notes in Math. 179, Springer, Berlin, 1971, 95-110. [59] W. Tautz, J. Top, and A. Verberkmoes, Explicit hyperelliptic curves with real multiplication and permutation polynomials, Canad. J. Math. 43 (1991), 1055-1064. [60] J. Voight, Identifying the matrix ring: algorithms for quaternion algebras and quadratic forms, in: Quadratic and Higher Degree Forms, Dev. Math. 31, Springer, New York, 2013, 255-298. [61] P. van Wamelen, Examples of genus two CM curves defined over the rationals, Math. Comp. 68 (1999), 307-320.

[33] H. Weber, Theorie der Abelschen Functionen vom Geschlecht 3, Reimer, Berlin, 1876.

[34] A. Weng, A class of hyperelliptic CM-curves of genus three, J. Ramanujan Math. Soc. 16 (2001), 339-372. · Zbl 1066.11028