Lee, Jungyun; Lee, Yoonjin

Regulators and class numbers of an infinite family of quintic function fields. (English)

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The objective of this paper is the study of a certain infinite family $\{K_h\}_{h\in A}$ of quintic function fields assuming that the characteristic p is different from 5, where $A = \mathbb{F}_q[x]$ and $k = \mathbb{F}_q(x)$. In fact, the elements of the family $\{K_h\}$ are subfields of cyclotomic function fields, that have the same conductors. The authors find the system of fundamental units and regulators of the elements of $\{K_h\}$ (Theorem 1.1), obtaining a result on the divisibility of the class numbers of cyclotomic function fields (Theorem 1.2). In fact they find the ideal class number $h(\mathcal{O}_h)$ of K_h (Theorem 1.3).

One of the main tools is the use of the notion of Lagrange resolvents of the generating quintic polynomials $F_h(x)$ of K_h. From the Lagrange resolvents, it is determined the rank of the unit group of K_h. This unit rank is 4 and in fact, K_h is a totally real function field. The regulator and the system of fundamental units of K_h are explicitly found. In fact the regulator R_h of K_h equals $R_h = 71(\deg h)^4$.

In the last section, it is shown that there are infinitely many irregular primes of second class $f \in A$ such that $h(k(\Lambda_f)^+) \equiv 0 \bmod p^4$ where $k(\Lambda_N)^+$ denotes the real subfield of the cyclotomic function field $k(\Lambda_N)$, $N \in A$.

Reviewer: Gabriel D. Villa Salvador (Ciudad de México)

MSC:

11R60 Cyclotomic function fields (class groups, Bernoulli objects, etc.)
11R29 Class numbers, class groups, discriminants
11R58 Arithmetic theory of algebraic function fields

Keywords:

regulator; function field; quintic extension; class number; cyclotomic function field; irregular prime; totally real cyclotomic function fields; ideal class numbers; Lagrange resolvents

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