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Nonuniform structures in solutions to the Euler equations. (English) Zbl 0695.76002

Rend. Semin. Mat., Torino Fasc. Spec., 87-94 (1988).

We are concerned with the incompressible Euler equations in two space dimensions,

$$(1)\quad \partial_t v + divv \otimes v + \nabla p = 0, \quad divv = 0, \quad x \in R^2, \quad t > 0,$$

specifically with solution sequences v_ϵ having locally bounded kinetic energy

$$(2)\quad \int_\Omega |v_\epsilon(x,t)|^2 dx \leq c,$$

for all compact subsets Ω on R^2, where the constant c may depend on Ω but not on the index ϵ.

This subject is part of a program that deals with the representation and analysis of nonuniform structures in solution sequences to the Euler equations in two and three space dimensions and other nonlinear conservative systems. In general, nonuniform behaviour in solution sequences may manifest itself in a variety of ways, for example, through the persistence of oscillations and the development of concentration. Here we be concerned primarily with the structure of concentrations in solution sequences v_ϵ satisfying (2).

As a consequence of the uniform energy bound (2) we may assume by passing to a subsequence, that v_ϵ converges weakly to a vector field v in L^2 of compact subsets of physical space $R^2 \times R^+$, i.e. in

$$L^2_{loc}(R^2 \times R^+): \quad \lim_{\epsilon \to 0} \iint \Phi v_\epsilon dxdt = \iint \Phi v dxdt$$

for all Φ in $C_0^\infty(R^2 \times R^+)$. Thus, the local average of v_ϵ converges to the local average of v. In this context we are concerned with obstructions to strong L^2 convergence, specifically the existence and size of exceptional sets E such that $\lim \iint_E |v_\epsilon - v|^2 dxdt \neq 0$ and with the associated response of the nonlinear inertial terms $v_\epsilon \otimes v_\epsilon$ to such losses of strong convergence.

Two of our main results are the following. The first provides an upper bound on the Hausdorff dimension of exceptional sets on which L^2 compactness may be lost, i.e. on which kinetic energy may concentrate in an L^2 weakly convergent 2-D Euler sequence v_ϵ. The second establishes a surprising robust quality of the inertial terms $v \otimes v$ acting on such sequences, namely, complete insensitivity to concentrations in the energy field. The latter property expresses itself analytically through a process of concentration cancellation and physically through a process of local energy equipartition.

One of the steps in our program of assessing energy concentrations involves the analysis of a reduced defect measure θ which is associated with an arbitrary L^2 weakly convergent sequence ($v_\epsilon \to v$) through the equation

$$(4)\quad \theta(E) = \limsup \iint_E |v_\epsilon - v|^2 dxdt,$$

where E is a Borel subset of $R^2 \times R^+$. The set function θ is a nonnegative finitely subadditive outer measure which vanishes precisely on those sets E where v_ϵ converges strongly to v: θ is concentrated on the L^2 exceptional sets of the sequence v_ϵ.

One of the objectives of our program is to prove that the reduced defect measure is concentrated on a small set. To this end we establish the following theorem: Suppose v_ϵ is a 2-D Euler sequence defined on the strip $[0,T]$ with uniformly bounded energy and vorticity in the sense that

$$\iint_{R^2} |v_\epsilon(x,t)|^2 dx \leq c,$$

total mass $\omega_\epsilon(\cdot,t) \leq c$, where the constant is independent of ϵ and t. Then there exists an L^2 weakly convergent subsequence for which the associated reduced defect measure is concentrated inside a set with

Hausdorff dimension less than one in the sense of finitely subadditive outer measures.

A corresponding local statement holds for solution sequences defined on a ball in $R^2 \times R^+$.

The result is sharp from the viewpoint of Hausdorff dimension. The aforementioned examples of steady and rotating vortices provide explicit sequences for which θ is concentrated on a one-dimensional set, in the sense of finitely subadditive outer measures.

We say that a finitely subadditive outer measure θ is concentrated inside a set E if E^c is the union of a countable number of null sets of θ. In general θ is not countably subadditive. If it were then E^c would also be a null set.

MSC:

76A02 Foundations of fluid mechanics
35Q30 Navier-Stokes equations
76B47 Vortex flows for incompressible inviscid fluids

Keywords:

incompressible Euler equations in two space dimensions; solution sequences; locally bounded kinetic energy; analysis of nonuniform structures in solution sequences; nonlinear conservative systems; development of concentration; upper bound on the Hausdorff dimension of exceptional sets; process of concentration cancellation; local energy equipartition; reduced defect measure; weakly convergent sequence; nonnegative finitely subadditive outer measure; vorticity