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Factorizations and Hardy-Rellich-type inequalities. (English) Zbl 1402.35014

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Summary: The principal aim of this note is to illustrate how factorizations of singular, even-order partial differential operators yield an elementary approach to classical inequalities of Hardy-Rellich-type. More precisely, introducing the two-parameter n-dimensional homogeneous scalar differential expressions $T_{\alpha,\beta} := -\Delta + \alpha|x|^{-2}x\cdot\nabla + \beta|x|^{-2}$, $\alpha,\beta \in \mathbb{R}$, $x \in \mathbb{R}^n\setminus\{0\}$, $n \in \mathbb{N}$, $n \geq 2$, and its formal adjoint, denoted by $T^+_{\alpha,\beta}$, we show that nonnegativity of $T^+_{\alpha,\beta}T_{\alpha,\beta}$ on $C_0^\infty(\mathbb{R}^n\setminus\{0\})$ implies the fundamental inequality

$$\begin{aligned}\int_{\mathbb{R}^n}[(\Delta f)(x)]^2\,d^nx \geq [(n-4)\alpha-2\beta]&\int_{\mathbb{R}^n}|x|^{-2}|(\nabla f)(x)|^2\,d^nx\\ -\alpha(\alpha-4)&\int_{\mathbb{R}^n}|x|^{-4}|x\cdot(\nabla f)(x)|^2\,d^nx\\ +\beta[(n-4)(\alpha-2)-\beta]&\int_{\mathbb{R}^n}|x|^{-4}|f(x)|^2\,d^nx,\\ f\in C_0^\infty(\mathbb{R}^n\setminus\{0\}).&\end{aligned}$$

A particular choice of values for α and β yields known Hardy-Rellich-type inequalities, including the classical Rellich inequality and an inequality due to Schmincke. By locality, these inequalities extend to the situation where $\mathbb{R}^n$ is replaced by an arbitrary open set $\Omega \subseteq \mathbb{R}^n$ for functions $f \in C_0^\infty(\Omega\setminus\{0\})$.

Perhaps more importantly, we will indicate that our method, in addition to being elementary, is quite flexible when it comes to a variety of generalized situations involving the inclusion of remainder terms and higher-order operators.

For the entire collection see [Zbl 1390.35006].

MSC:

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35J75	Singular elliptic equations

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