Matala-aho, Tapani; Seppälä, Louna
Hermite-Thue equation: Padé approximations and Siegel's lemma. (English) Zbl 1436.11084
J. Number Theory 191, 345-383 (2018).

In Diophantine approximation the use of Padé approximants is used to proving sharp transcendence measures. Specifically so-called *twin* approximations can successfully used to prove Baker-type transcendence measures for e.

Let $l_1,\dots,l_m$ be positive integers, let $\alpha_1,\dots,\alpha_m$ be distinct variables. Denote $\overline{\alpha}=(\alpha_1,\dots,\alpha_m)^T$, $\bar{l}=(l_1,\dots,l_m)^T$ and $L=l_1+\ldots+l_m$. The *twin problem* for the m-tuple of exponentials $e^{\alpha_j t}\}_{j=1}^m$ is then given by:

Find an explicit polynomial $B_{\bar{l},0}(t,\overline{\alpha})$, polynomials $B_{\bar{l},j}(t,\overline{\alpha})$ and remainders $S_{\bar{l},j}(t,\overline{\alpha})$, $j=1,\dots,m$ satisfying

$$B_{\bar{l},0}(t,\overline{\alpha})e^{\alpha_j t}-B_{\bar{l},j}(t,\overline{\alpha})=S_{\bar{l},j}(t,\overline{\alpha}),\ j=1,\dots,m$$

and

$$\deg B_{\bar{l},j}(t,\overline{\alpha})\le L, 0\le j\le m;\ L+l_j+1\le \operatorname{ord} S_{\bar{l},j}(t,\overline{\alpha})<\infty,\ 1\le j\le m.$$

Moreover, the authors use the *wild* version of the *tame*, where the l_j and $\bar{l}$, used in the definition of the B's, are replaced by ν_j and $\overline{\nu}$:

$$\overline{\nu}=(\nu_1,\dots,\nu_m)^T\in\mathbb{Z}^m_{\ge 1},\ \nu_1\le l_1,\dots,\nu_m\le l_m,\nu_1+\ldots+\nu_m=:M\le L$$

The main results read as follows:

Theorem 2.1. Let $\alpha_1,\dots,\alpha_m\in\mathbb{Z}$. Then

$$\left(\prod_{1\le j\le m} a_j^{\frac{\nu_j}{2}}\right)\prod_{1\le i<j\le m}(a_i-a_j)^{\min(\nu_i^2,\nu_j^2)}|_{\mathbb{Z}}D(\overline{a}),$$

where $D(\overline{\alpha})$ is the greatest common divisor of all the $M\times M$ minors of the matrix $\mathcal{V}(\overline{a})\in\mathcal{M}_{M\times(L+1)}(\mathbb{Z})$.

Theorem 2.2. Let $l_1,\dots,l_m$ be positive integers and let $\alpha_1,\dots,\alpha_m$ be distinct variables. Denote $\overline{\alpha}=(\alpha_1,\dots,\alpha_m)^T$, $\bar{l}=(l_1,\dots,l_m)^T$ and $L=l_1+\ldots+l_m$. Then the twin problem for the exponentials, defined above, has a non-zero solution of polynomials $B_{\bar{l},j}\in\mathbb{Q}[t,\overline{\alpha}]$. Moreover

$$B_{\bar{l},0}(t,\overline{\alpha})=\sum_{i=0}^{L}\frac{L!}{i!}\tau_i(\bar{l},\overline{\alpha})t^i,\ \tau_i(\bar{l},\overline{\alpha})=\frac{(-1)^i\mathcal{V}[i]}{T(\bar{l},\overline{\alpha})}\in\mathbb{Z}[\overline{\alpha}],$$

where

$$T(\bar{l},\overline{\alpha}):=\alpha_1^{\frac{l_1}{2}}\cdots\alpha_m^{\frac{\alpha_m}{2}}\prod_{1\le i<j\le m}(\alpha_i-\alpha_j)^{\min\{l_i^2,l_j^2\}}$$

is a common factor of the $L\times L$ minors $\mathcal{V}[i]$, $i=0,\dots,L$, of the matrix $\mathcal{V}(\overline{\alpha})$.

The layout of the paper is as follows:

Reviewer: Marcel G. de Bruin (Heemstede)

MSC:

11J81 Transcendence (general theory)
41A21 Padé approximation
11C20 Matrices, determinants in number theory

Keywords:

Diophantine approximation; Padé approximation; Siegel's lemma; Vandermonde-type determinant

Full Text: DOI

References:

[1] Baker, A., On some Diophantine inequalities involving the exponential function, Canad. J. Math., 17, 616-626, (1965) · Zbl 0147.30901

[2] Baker, A., Transcendental number theory, (1975), Cambridge University Press · Zbl 0297.10013

[3] Baker, G.; Graves-Morris, P. R., Padé approximants, (1996), Cambridge University Press · Zbl 0923.41001

[4] Bombieri, E.; Vaaler, J. D., On Siegel's lemma, Invent. Math., 73, 11-32, (1983) · Zbl 0533.10030

[5] de Bruin, M. G., Some aspects of simultaneous rational approximation, (Numerical Analysis and Mathematical Modelling, Banach Center Publ., vol. 24, (1990), PWN Warsaw), 51-84

[6] Ernvall-Hytönen, A-M.; Leppälä, K.; Matala-aho, T., An explicit Baker-type lower bound of exponential values, Proc. Roy. Soc. Edinburgh Sect. A, 145, 1153-1182, (2015) · Zbl 1387.11059

[7] Ernvall-Hytönen, A-M.; Matala-aho, T.; Seppälä, L., On Mahler's transcendence measure for \textit{e}, Constr. Approx.,

(2018)

[8] Fel'dman, N. I.; Nesterenko, Yu. V., Number theory IV, Encyclopaedia Math. Sci., vol. 44, (1998), Springer Berlin

[9] Flowe, R. P.; Harris, G. A., A note on generalized Vandermonde determinants, SIAM J. Matrix Anal. Appl., 14, 1146-1151, (1993) · Zbl 0789.15004

[10] Fukshansky, L., Siegel's lemma with additional conditions, J. Number Theory, 120, 13-25, (2006) · Zbl 1192.11018

[11] Hata, M., Remarks on Mahler's transcendence measure for \textit{e}, J. Number Theory, 54, 81-92, (1995) · Zbl 0839.11027

[12] Hermite, C., Sur la fonction exponentielle, C. R. Acad. Sci. Paris, 77, (1873) · Zbl 05.0248.01

[13] Huttner, M., Constructible sets of linear differential equations and effective rational approximations of polylogarithmic functions, Israel J. Math., 153, 1-43, (2006) · Zbl 1143.34057

[14] Krattenthaler, C., Advanced determinant calculus, The Andrews Festschrift, Maratea, 1998, Sém. Lothar. Combin., 42, (1999), Art. B42q, 67 pp. · Zbl 0923.05007

[15] Mahler, K., On a paper by A. Baker on the approximation of rational powers of \textit{e}, Acta Arith., 27, 61-87, (1975) · Zbl 0305.10030

[16] Matala-aho, T., Type II Hermite-Padé approximations of generalized hypergeometric series, Constr. Approx., 33, 289-312, (2011) · Zbl 1236.41017

[17] Matala-aho, T., A geometric face of Diophantine analysis, (Diophantine Analysis, Trends Math., (2016), Springer), 129-174 · Zbl 1416.11121

[18] Nesterenko, Yu. V., Hermite-Padé approximants of generalised hypergeometric series, Russian Acad. Sci. Sb. Math., 83, I, (1995)

[19] Nikišin, E. M., On simultaneous Padé approximants, Math. USSR, Sb., 41, 4, (1982) · Zbl 0478.30007

[20] van der Poorten, A., Some determinants that should be better known, J. Aust. Math. Soc. A, 21, 278-288, (1976) · Zbl 0343.15002

[21] Rotman, J., Advanced modern algebra, (2002), Prentice Hall · Zbl 0997.00001

[22] Shidlovskiĭ, A. B., Transcendental numbers, (1989), Walter de Gruyter · Zbl 0689.10043

[23] Siegel, C. L., Über einige anwendungen diophantischer approximationen, Abh. Preuss. Akad. Wiss. Phys.-Math. Kl., 1, 1-70, (1929) · Zbl 56.0180.05

[24] Siegel, C. L., Transcendental numbers, Ann. of Math. Stud., vol. 16, (1949), Princeton University Press Princeton, N.J. · Zbl 0039.04402

[25] Thue, A., Om en generel i store Hele tal uløsbar ligning, Nor. Vidensk. Selsk. Skr., 7, 1-15, (1908)

[26] Thue, A., Über annäherungswerte algebraischer zahlen, J. Reine Angew. Math., 135, 284-305, (1909) · Zbl 40.0265.01