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**Hermite-Thue equation: Padé approximations and Siegel's lemma.** (English) Zbl 1436.11084  
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In Diophantine approximation the use of Padé approximants is used to proving sharp transcendence measures. Specifically so-called *twin* approximations can successfully used to prove Baker-type transcendence measures for  $e$ .

Let  $l_1, \dots, l_m$  be positive integers, let  $\alpha_1, \dots, \alpha_m$  be distinct variables. Denote  $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)^T$ ,  $\bar{l} = (l_1, \dots, l_m)^T$  and  $L = l_1 + \dots + l_m$ . The *twin problem* for the  $m$ -tuple of exponentials  $e^{\alpha_j t}$   $\}_{j=1}^m$  is then given by:

Find an explicit polynomial  $B_{\bar{l},0}(t, \bar{\alpha})$ , polynomials  $B_{\bar{l},j}(t, \bar{\alpha})$  and remainders  $S_{\bar{l},j}(t, \bar{\alpha})$ ,  $j = 1, \dots, m$  satisfying

$$B_{\bar{l},0}(t, \bar{\alpha})e^{\alpha_j t} - B_{\bar{l},j}(t, \bar{\alpha}) = S_{\bar{l},j}(t, \bar{\alpha}), \quad j = 1, \dots, m$$

and

$$\deg B_{\bar{l},j}(t, \bar{\alpha}) \leq L, 0 \leq j \leq m; \quad L + l_j + 1 \leq \text{ord } S_{\bar{l},j}(t, \bar{\alpha}) < \infty, \quad 1 \leq j \leq m.$$

Moreover, the authors use the *wild* version of the *tame*, where the  $l_j$  and  $\bar{l}$ , used in the definition of the  $B$ 's, are replaced by  $\nu_j$  and  $\bar{\nu}$ :

$$\bar{\nu} = (\nu_1, \dots, \nu_m)^T \in \mathbb{Z}_{\geq 1}^m, \quad \nu_1 \leq l_1, \dots, \nu_m \leq l_m, \quad \nu_1 + \dots + \nu_m =: M \leq L$$

The main results read as follows:

Theorem 2.1. Let  $\alpha_1, \dots, \alpha_m \in \mathbb{Z}$ . Then

$$\left( \prod_{1 \leq j \leq m} a_j^{\frac{\nu_j}{2}} \right) \prod_{1 \leq i < j \leq m} (a_i - a_j)^{\min(\nu_i, \nu_j)} |_{\mathbb{Z}D(\bar{\alpha})},$$

where  $D(\bar{\alpha})$  is the greatest common divisor of all the  $M \times M$  minors of the matrix  $\mathcal{V}(\bar{\alpha}) \in \mathcal{M}_{M \times (L+1)}(\mathbb{Z})$ .

Theorem 2.2. Let  $l_1, \dots, l_m$  be positive integers and let  $\alpha_1, \dots, \alpha_m$  be distinct variables. Denote  $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)^T$ ,  $\bar{l} = (l_1, \dots, l_m)^T$  and  $L = l_1 + \dots + l_m$ . Then the twin problem for the exponentials, defined above, has a non-zero solution of polynomials  $B_{\bar{l},j} \in \mathbb{Q}[t, \bar{\alpha}]$ . Moreover

$$B_{\bar{l},0}(t, \bar{\alpha}) = \sum_{i=0}^L \frac{L!}{i!} \tau_i(\bar{l}, \bar{\alpha}) t^i, \quad \tau_i(\bar{l}, \bar{\alpha}) = \frac{(-1)^i \mathcal{V}[i]}{T(\bar{l}, \bar{\alpha})} \in \mathbb{Z}[\bar{\alpha}],$$

where

$$T(\bar{l}, \bar{\alpha}) := \alpha_1^{\frac{l_1}{2}} \cdots \alpha_m^{\frac{l_m}{2}} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{l_i^2, l_j^2\}}$$

is a common factor of the  $L \times L$  minors  $\mathcal{V}[i]$ ,  $i = 0, \dots, L$ , of the matrix  $\mathcal{V}(\bar{\alpha})$ .

The layout of the paper is as follows:

§1. Introduction

1.1 Hermite-Padé approximation

1.2 The twin problem

1.3 Siegel's lemma

§2. Results

The Theorems 2.1 and 2.2

§3. Preliminaries and tools

3.1 Exterior algebras  
3.1.1 Increasing lists  
3.1.2 Increasing vectors  
3.1.3 Grassmann coordinates  
3.2 Generalised minor expansions  
3.3 Polynomial rings  
§4. Genralised Vandermonde-type polynomial block matrices  
4.1 Case A  
4.2 Case B  
§5. Hermite-Padé approximations to the exponential function: tame case  
5.1 A new proof of Theorem 5.1  
5.1.1 Factors of  $\mathcal{U}[0]$   
5.1.2 Rank  
5.1.3 Cramer's rule  
§6. Hermite-Padé approximations to the exponential function: wild case  
6.1 The twin problem  
6.2 Siegel's lemma  
6.3 The Bombieri-Vaaler version of Siegel's lemma  
6.4 Common factor  
Leads to the proof of Theorem 2.1  
6.5 Rank  
6.6 Twin type II Padé approximants  
Leads to the proof of Theorem 2.2  
Appendix A. Some examples of the matrix  $\mathcal{V}(\bar{\alpha})$   
References (26 items)

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