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Global solvability and global hypoellipticity in Gevrey classes for vector fields on the torus.

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Let $L = \frac{\partial}{\partial t} + \sum_{j=1}^{N}(a_j + ib_j)(t)\frac{\partial}{\partial x_j}$ be a vector field defined on $\mathbb{T}^{N+1} \simeq \mathbb{R}^{N+2}/2\pi\mathbb{Z}^{N+1}$, where a_j, b_j are real-valued functions and belonging to the Gevrey class $G^s(\mathbb{T}^1)$, $s > 1$, for $j = 1, \ldots, N$. The author presents a complete characterization for the s-global solvability and s-global hypoellipticity of L. His results are linked to Diophantine properties of the coefficients and, also, connectedness of certain sublevel sets.

For $s > 1$ introduce the following exponential Diophantine conditions for a pair $(\alpha, \beta) \in \mathbb{R}^N \times \mathbb{R}^N$:

$(EDC)_1^s$: For each $\varepsilon > 0$ there exists a positive constant C_ε such that

$$|\tau + \langle \xi, \alpha + i\beta\rangle| \geq C_\varepsilon \exp\{-\varepsilon(|\xi| + |\tau|)^{1/s}\},$$

for all $(\xi, \tau) = (\xi_1, \ldots, \xi_N, \tau) \in \mathbb{Z}^{N+1} \setminus \{0\}$.

$(EDC)_2^s$: For each $\varepsilon > 0$ there exists a positive constant C_ε such that

$$|\tau + \langle \xi, \alpha + i\beta\rangle| \geq C_\varepsilon \exp\{-\varepsilon|\xi| + |\tau|)^{1/s}\},$$

for all $(\xi, \tau) = (\xi_1, \ldots, \xi_N, \tau) \in \mathbb{Z}^{N+1}$ such that $\tau + \langle \xi, \alpha + \beta\rangle \neq 0$.

Define $\alpha = (a_1, \ldots, a_N)$, $\beta = (\beta_1, \ldots, \beta_N)$, $a_{j,0} = \frac{1}{2\pi}\int_0^{2\pi} a_j(t)\,dt$, $b_{j0} = \frac{1}{2\pi}\int_0^{2\pi} a_j(t)\,dt$, $\alpha_0 = (a_{10}, \ldots, a_{N0})$, $\beta_0 = (b_{10}, \ldots, b_{N0})$. The author has obtained the following two theorems.

Theorem 1. Let L be given by above. Then, L is s-global solvable ($s > 1$) if and only if one of the following situations occurs:

(I) For each $j = 1, \ldots, N$, b_j vanishes identically, and $(\alpha_0, 0)$ satisfies $(EDC)_2^s$.

(II) At least one b_j dones not vanishes identically, $b_{k0} = 0$ for each $k = 1, \ldots, N$, $\alpha_0 \in \mathbb{Z}^N$, and the sublevel sets

$$\left\{ t \in \mathbb{T}^1;\ \int_0^t (\xi, \beta(\tau))\,d\tau < r \right\},\ r \in \mathbb{R},\ \xi \in \mathbb{Z}^N,$$

are connected.

(III) $b_{j0} \neq 0$ for at least one $j \in \{1, \ldots, N\}$, and the following conditions hold:

(III.1) $\dim \operatorname{span}\{b_1, \ldots, b_N\} = 1$;

(III.2) the functions b_j do not change sign;

(III.3) the pair (α_0, β_0) satisfies $(EDC)_2^s$.

Theorem 2. Let L be given by above. Then, L is s-global hypoelliptic ($s > 1$) if and only if the following conditions are satisfied:

(1) each b_j does not change sign;

(2) $\dim \operatorname{span}\{b_1, \ldots, b_N\} \leq 1$;

(3) (α_0, β_0) satisfies $(EDC)_1^s$.

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MSC:

35H10 Hypoelliptic equations
35A01 Existence problems for PDEs: global existence, local existence, non-existence
35B10 Periodic solutions to PDEs
35B65 Smoothness and regularity of solutions to PDEs

Keywords:

Gevrey solvability; Gevrey hypoellipticity; Fourier series; Diophantine properties of the coefficients

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