Böhm, Janko; **Bringmann, Kathrin**; **Buchholz, Arne**; **Markwig, Hannah**
Tropical mirror symmetry for elliptic curves. (English) Zbl 1390.14191
J. Reine Angew. Math. 732, 211-246 (2017); erratum ibid. 760, 163-164 (2020).

Elliptic curves provide the first non-trivial examples of Calabi-Yau varieties to check the conjectural predictions of mirror symmetry, a phenomenon which arose from dualities in string theories. Mirror symmetry for elliptic curves has been extensively studied in the mathematics literature. The paper provides a tropical generalization to the classically known results for elliptic curves in this context.

Using tropical geometric techniques to approach mirror symmetry lies in the heart of the program initiated by Mark Gross and Bernd Siebert. In his paper [Adv. Math. 224, No. 1, 169–245 (2010; Zbl 1190.14038)], *M. Gross* used various ideas of the Gross-Siebert program to explain tropically mirror symmetry for $\mathbb{P}^2$, where the mirror is $(\mathbb{C}^\times)^2$, endowed with a suitable potential function. The paper under review can be seen as a sequel of this tropical approach. However, it provides the first tropical proof of the mirror symmetry conjecture in the compact Calabi-Yau case. Some techniques developed in the paper are also inspired by various other works, in particular by [*E. Goujard* and *M. Moeller*, "Counting Feynman-like graphs: quasimodularity and Siegel-Veech weight", Preprint, arXiv:1609.01658], while carrying out computations for Feynmann integrals.

The mirror symmetry conjecture, in particular, implies a relationship between the generating series of Hurwitz numbers of the elliptic curve to Feynman integrals. The main results of this paper investigate this classical relationship on a tropical level. Moreover, a tropical correspondence theorem for Hurwitz numbers, relating tropical and classical Hurwitz invariants in all genera and degrees is shown.

The tropical methods developed have several advantages. They provide computational accessibility to obtain the involved invariants in mirror symmetry. Throughout the paper, several computations are carried out in various explicit examples. Moreover, under suitable tropical correspondence theorems that are provided, the tropical statements actually lead us to proofs of the classical mirror symmetry predictions.

Reviewer: Hulya Arguz (London)

MSC:

14T20	Geometric aspects of tropical varieties
14J33	Mirror symmetry (algebro-geometric aspects)

Keywords:
tropical mirror symmetry; elliptic curves; Hurwitz numbers

Software:
ellipticcovers.lib; SINGULAR

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