A Lefschetz fibration on a closed, smooth, oriented 4-manifold $X$ is a smooth surjective map $f : X \to S^2$ whose critical locus consists of finitely many points $p_i$, such that at $p_i$ and at $f(p_i)$ there are local complex coordinates with respect to which $f$ takes the form $(z_1, z_2) \mapsto z_1 z_2$. The pair $(X, f)$ is called a genus-$g$ Lefschetz fibration for the genus $g$ of a regular fiber $F$ of $f$. A common way to construct new Lefschetz fibrations is the fiber sum operation. If $(X_i, f_i)$ is a genus-$g$ Lefschetz fibration with regular fiber $F_i$ for $i = 1, 2$, then the fiber sum is a genus-$g$ Lefschetz fibration $f$ on $X = (X_1, F_1)\# \varphi(X_2, F_2)$, obtained by removing a fibered tubular neighborhood of each $F_i$ and then identifying the resulting boundaries via complex conjugation on $S^1$ times a chosen orientation preserving diffeomorphism $\varphi : F_1 \to F_2$. A Lefschetz fibration $(X, f)$ is said to be indecomposable if it cannot be expressed as a fiber sum.

A Lefschetz fibration $(X, f)$ is called minimal if there are no exceptional spheres contained in the fibers. If $\Sigma_g^m$ is a compact, connected, oriented surface of genus $g$ with $m$ boundary components, then the group $\Gamma^m_g$, composed of orientation-preserving self-homeomorphisms of $\Sigma_g^m$ which restrict to the identity along $\partial \Sigma_g^m$, as well, is called the mapping class group. If $t_c \in \Gamma^m_g$ is the positive Dehn twist along the simple closed curve $c \subset \Sigma_g^m$, $\{c_i\}$ is a nonempty collection of simple closed curves on $\Sigma_g^m$, $\{\delta_j\}$ is a collection of $m$ curves parallel to distinct boundary components of $\Sigma_g^m$, $\{k_j\}$ is a collection of $m$ integers, and the relation $t_{c_1} \cdots t_{c_l} t_{c_1} = t_{k_1} \cdots t_{k_m} t_{k_1}$ holds in $\Gamma^m_g$, then $t_{c_1} \cdots t_{c_l} t_{c_1}$ is called a positive factorization of length $l$ of the mapping class $t_{k_1} \cdots t_{k_m}$ in $\Gamma^m_g$. Existence of minimal symplectic structures on 4-manifolds is a fundamental question in smooth 4-manifold topology. There has been much interest in producing minimal symplectic 4-manifolds in the homeomorphism classes of standard simply-connected 4-manifolds with small second homology, such as blow-ups of $\mathbb{CP}^2$ or $3\mathbb{CP}^2$.

In this paper, the authors demonstrate ways to construct positive factorization for Lefschetz fibrations with small number of critical points, as they correspond to 4-manifolds with small second homology which allow to provide simple descriptions of many new small exotic 4-manifolds. The authors show that there exist decomposable genus-2 Lefschetz fibrations whose total spaces are minimal symplectic 4-manifolds homeomorphic but not diffeomorphic to complex rational surfaces $\mathbb{CP}^2\#p\overline{\mathbb{CP}}^2$ for $p = 7, 8, 9$, and to $3\mathbb{CP}^2\#q\overline{\mathbb{CP}}^2$ for $q = 12, \ldots, 19$, and describe all the genus-2 Lefschetz fibrations explicitly via positive factorizations in the mapping class group of a genus-2 surface with one boundary component. They also show that any simply-connected minimal genus-2 Lefschetz fibration $(X, f)$ with $b^+ (X) \leq 3$ is homeomorphic to $\mathbb{CP}^2\#p\overline{\mathbb{CP}}^2$ for some $7 \leq p \leq 9$, or to $3\mathbb{CP}^2\#q\overline{\mathbb{CP}}^2$ for $12 \leq q \leq 19$. Finally, they prove that there exist decomposable minimal genus-2 Lefschetz fibrations over $T^2$ and $\sigma_2$ which are equivalent via Luttinger surgeries to minimal symplectic 4-manifolds $\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$, respectively.

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MSC:

57R17 Symplectic and contact topology in high or arbitrary dimension
57R57 Applications of global analysis to structures on manifolds
53D35 Global theory of symplectic and contact manifolds

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