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Matrix rank and inertia formulas in the analysis of general linear models. (English)

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Summary: Matrix mathematics provides a powerful tool set for addressing statistical problems, in particular, the theory of matrix ranks and inertias has been developed as effective methodology of simplifying various complicated matrix expressions, and establishing equalities and inequalities occurred in statistical analysis. This paper describes how to establish exact formulas for calculating ranks and inertias of covariances of predictors and estimators of parameter spaces in general linear models (GLMs), and how to use the formulas in statistical analysis of GLMs. We first derive analytical expressions of best linear unbiased predictors/best linear unbiased estimators (BLUPs/BLUEs) of all unknown parameters in the model by solving a constrained quadratic matrix-valued function optimization problem, and present some well-known results on ordinary least-squares predictors/ordinary least-squares estimators (OLSPs/OLSEs). We then establish some fundamental rank and inertia formulas for covariance matrices related to BLUPs/BLUEs and OLSPs/OLSEs, and use the formulas to characterize a variety of equalities and inequalities for covariance matrices of BLUPs/BLUEs and OLSPs/OLSEs. As applications, we use these equalities and inequalities in the comparison of the covariance matrices of BLUPs/BLUEs and OLSPs/OLSEs. The work on the formulations of BLUPs/BLUEs and OLSPs/OLSEs, and their covariance matrices under GLMs provides direct access, as a standard example, to a very simple algebraic treatment of predictors and estimators in linear regression analysis, which leads a deep insight into the linear nature of GLMs and gives an efficient way of summarizing the results.

MSC:

15A03 Vector spaces, linear dependence, rank, lineability
15A09 Theory of matrix inversion and generalized inverses
62H12 Estimation in multivariate analysis
62J05 Linear regression; mixed models

Keywords:

general linear model; predictor; estimator; covariance matrix; rank; inertia; equality; inequality

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