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Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$. (English) [Zbl 1357.82011](#)
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Summary: This article studies the planar Potts model and its random-cluster representation. We show that the phase transition of the nearest-neighbor ferromagnetic q -state Potts model on \mathbb{Z}^2 is continuous for $q \in \{2, 3, 4\}$, in the sense that there exists a unique Gibbs state, or equivalently that there is no ordering for the critical Gibbs states with monochromatic boundary conditions.

The proof uses the random-cluster model with cluster-weight $q \geq 1$ (note that q is not necessarily an integer) and is based on two ingredients:

- The fact that the two-point function for the free state decays sub-exponentially fast for cluster-weights $1 \leq q \leq 4$, which is derived studying parafermionic observables on a discrete Riemann surface.
- A new result proving the equivalence of several properties of critical random-cluster models:
 - the absence of infinite-cluster for wired boundary conditions,
 - the uniqueness of infinite-volume measures,
 - the sub-exponential decay of the two-point function for free boundary conditions,
 - a Russo-Seymour-Welsh type result on crossing probabilities in rectangles with arbitrary boundary conditions.

The result has important consequences toward the study of the scaling limit of the random-cluster model with $q \in [1, 4]$. It shows that the family of interfaces (for instance for Dobrushin boundary conditions) are tight when taking the scaling limit and that any sub-sequential limit can be parametrized by a Loewner chain. We also study the effect of boundary conditions on these sub-sequential limits. Let us mention that the result should be instrumental in the study of critical exponents as well.

MSC:

- 82B20** Lattice systems (Ising, dimer, Potts, etc.) and systems on graphs arising in equilibrium statistical mechanics
- 60K35** Interacting random processes; statistical mechanics type models; percolation theory
- 82B26** Phase transitions (general) in equilibrium statistical mechanics
- 82D40** Statistical mechanical studies of magnetic materials
- 82B27** Critical phenomena in equilibrium statistical mechanics

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[planar Potts model](#); [phase transition](#); [random-cluster model](#)

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