Dicks, Warren; **Dunwoody, M. J.**
Groups acting on graphs. (English) Zbl 0665.20001
Cambridge Studies in Advanced Mathematics, 17. Cambridge etc.: Cambridge University Press. xvi, 283 p. £30.00; $ 54.50 (1989).

The present work is an advanced text and research monograph devoted to some of the most interesting modern developments on the border between algebra and topology. The interplay between these two large branches of pure mathematics has proved to be extremely fruitful, with mutual benefit in many instances. In particular, the theory of groups acting on graphs has led to a significant revitalization of combinatorial group theory and to important progress in low-dimensional topology.

The authors are well known for their substantial contributions to the subject. In this book they not only attempt to give a systematic account of known results (most of the topics discussed here appear in book form for the first time), but they also include some recent original results which appear for the first time in print.

The first chapter contains an exposition of the Bass-Serre theory of groups acting on trees and graphs, along the main lines of *J.-P. Serre*'s lectures [Arbres, amalgames, SL_2 (Astérisque 46, 1977; Zbl 0369.20013); English version: Trees (Springer 1980)]. This includes the description of the basic concepts, proofs of the central results (the structure theorem for groups acting on trees and its converse, then the structure theorem for groups acting on connected graphs), a description of the most frequently encountered concrete examples, and some of the most important applications in combinatorial group theory: the Kurosh subgroup theorem, the Nielsen-Schreier theorem, the Schreier index formula, and the Grushko-Neumann theorem.

Chapter II starts with a characterization of trees in terms of vertices acting as functions on the edge set. Then the concept of the Boolean ring of a connected G-graph is introduced; a construction is given which associates with each connected graph and positive integer n, a G-tree which reflects the disconnection of the graph when n edges are removed. As an application of this construction, a result due to *H. D. Macpherson* [Combinatorica 2, 63-69 (1982; Zbl 0492.05036)] is derived, which characterizes the infinite finite-valency distance-transitive graphs.

Chapter III is devoted to one of the authors' most important original technical result: the almost stability theorem. This turns out to be a powerful tool which allows to derive far reaching extensions of known results. The theory traces its origins from several earlier results, like Stallings' ends theorem. The proof is long and technical. The full strength of the almost stability theorem is demonstrated in Chapter IV, where many of its applications are obtained. One of these provides a method for constructing a tree out of a derivation to a projective module; this is used in all subsequent arguments. The other applications include the determination of the groups of cohomological dimension one over an arbitrary ring (the Stallings-Swan theorem), a generalization of this result which provides a new characterization of projective augmentation modules, a description of the splitting of the augmentation ideal of a group induced by the augmentation ideal of a subgroup, a new characterization of arbitrary groups with more than one end, and a cohomological criterion for the accessibility of a finitely generated group.

Up to this point, the whole treatment occurred at the one-dimensional level on the topological side. From now on, the discussion turns to higher dimensional topics, with emphasis on dimensions two and three.

The quite long Chapter V is devoted to the theory of Poincaré duality groups. It includes many important results but its final aim is to provide a proof for the beautiful result which states that, in dimension two, the Poincaré duality groups are precisely the infinite surface groups. At the starting point, the (co)homology theory of modules is reviewed. The discussion which follows centers around the concepts of GD^n-group and PD^n-group (geometric resp. Poincaré duality group of dimension n) over a commutative ring k, and their relative versions, GD^n-pairs and PD^n-pairs over k. A characterization of PD^n-groups over k is obtained for $n = 0$ and 1, and a similar result is stated as a conjecture for $n = 2$. A purely algebraic argument shows that the infinite surface groups are PD^2-groups over k. Further results include Serre's extension theorem and a proof of the fact that the subgroups of infinite index in a PD^n-group over k have cohomological dimension over k at most n-1. On the other hand it is shown that PD-groups acting on trees generate PD-pairs. A relative version of Stallings' ends theorem is then derived; this appears to be a result of fundamental importance since its consequences lead to the final result: if k has characteristic zero, then the PD^2-groups over k are precisely the infinite surface groups.

The final Chapter VI discusses actions of groups on one-dimensional skeleta of the two-dimensional simplicial complexes. For complexes whose underlying polyhedra satisfy certain geometric properties, algebraic and topological consequences on the group actions are derived. The arguments are rather technical and rely on the consideration of certain geometric configurations inside the polyhedra. A first important result obtained is a proof of the accessibility of almost finitely presented groups. The other results concern properly embedded surfaces in three-manifolds. The geometric configurations used in this case are a generalization of Haken's normal surfaces. The main results obtained here are geometric combinatorial proofs for equivariant versions of the Loop Theorem and of the Sphere Theorem (these theorems had been previously proved by Meeks and Yau using analytic minimal surface techniques).

The monograph is a valuable contribution to the progress of the theory of groups acting on low-dimensional spaces; it is certainly destined to become a standard reference for all those intending to approach research in this field.

Reviewer: J.Weinstein

MSC:

20-02 Research exposition (monographs, survey articles) pertaining to group theory
20F05 Generators, relations, and presentations of groups
05C25 Graphs and abstract algebra (groups, rings, fields, etc.)
55-02 Research exposition (monographs, survey articles) pertaining to algebraic topology
57-02 Research exposition (monographs, survey articles) pertaining to manifolds and cell complexes
05C05 Trees
18G20 Homological dimension (category-theoretic aspects)
20J05 Homological methods in group theory
55U30 Duality in applied homological algebra and category theory (aspects of algebraic topology)
57M05 Fundamental group, presentations, free differential calculus
57M15 Relations of low-dimensional topology with graph theory
57M20 Two-dimensional complexes (manifolds) (MSC2010)
57M35 Dehn's lemma, sphere theorem, loop theorem, asphericity (MSC2010)
57N10 Topology of general 3-manifolds (MSC2010)
57Q91 Equivariant PL-topology
20F34 Fundamental groups and their automorphisms (group-theoretic aspects)

Keywords:

groups acting on graphs; combinatorial group theory; low-dimensional topology; Bass-Serre theory; groups acting on trees; Kurosh subgroup theorem; Nielsen-Schreier theorem; Schreier index formula; Grushko-Neumann theorem; distance-transitive graphs; almost stability theorem; Stallings' ends theorem; groups of cohomological dimension one; Stallings-Swan theorem; augmentation ideal; accessibility; finitely generated group; Poincaré duality groups; infinite surface groups; PD^n-groups; PD-pairs; actions of groups; two-dimensional simplicial complexes; finitely presented groups