Gesztesy, Fritz; Mitrea, Marius; Nenciu, Irina; Teschl, Gerald

**Decoupling of deficiency indices and applications to Schrödinger-type operators with possibly strongly singular potentials.** (English) Zbl 1357.35099

Adv. Math. 301, 1022-1061 (2016).

The study of deficiency indices of symmetric operators in Hilbert spaces is one of the central problems in spectral theory of unbounded linear operators. One of the main objects in the paper under review are closed symmetric realizations $H$ in $L^2(\mathbb{R}^n)$ of Schrödinger-type operators

$$(-\Delta + V) \restriction_{C_0^\infty(\mathbb{R}^n \setminus \Sigma)}$$

whose potential $V$ has a countable number of well-separated singularities on compact sets $\Sigma_j$, $j \in J$, of $n$-dimensional Lebesgue measure zero, with $J \subseteq \mathbb{N}$ an index set and $\Sigma = \bigcup_{j \in J} \Sigma_j$.

The central result in the paper under review is an abstract approach to the question of decoupling of deficiency indices. Then the authors apply it to the concrete case of Schrödinger type operators in $L^2(\mathbb{R}^n)$ and to second-order elliptic differential operators on $\mathbb{R}^n$. For example, it is proved that under certain assumptions on $V$ and $\Sigma$ the defect $\mathrm{Def}(H)$ of $H$ can be computed in terms of the individual defects $\mathrm{Def}(H_j)$ of $H_j$, where $H_j$ is a closed symmetric realization of $(-\Delta + V) \restriction_{C_0^\infty(\mathbb{R}^n \setminus \Sigma_j)}$. More specifically, the authors derive the results that the defect of $H$ can be computed as the sum of over the individual defects of $H_j$,

$$\mathrm{Def}(H) = \sum_{j \in J} \mathrm{Def}(H_j).$$

The possibility that one and hence both sides equal $\infty$ is included. Here

$$\mathrm{Def}(A) := \frac{n_+(A) + n_-(A)}{2},$$

and $n_\pm(A)$ are the standard deficiency indices of $A$. Notice that $\mathrm{Def}(A) = n_\pm(A)$ if $n_+(A) = n_-(A)$ (the latter holds, in particular, when $A$ is lower semibounded or commutes with a conjugation).

Finally, let us mention that applications to Schrödinger operators with multipolar inverse-square potentials allow to considerably improve the results obtained in [*V. Felli* et al., J. Funct. Anal. 250, No. 2, 265–316 (2007; Zbl 1222.35074)].

Reviewer: Aleksey Kostenko (Donetsk)

**MSC:**

| | |
|---|---|
| 35J10 | Schrödinger operator, Schrödinger equation |
| 35P05 | General topics in linear spectral theory for PDEs |
| 47B25 | Linear symmetric and selfadjoint operators (unbounded) |
| 81Q10 | Selfadjoint operator theory in quantum theory, including spectral analysis |



**Keywords:**

Schrödinger operator; strongly singular potentials; deficiency indices; essential self-adjointness

**Full Text:** DOI arXiv

**References:**

[1] Akhiezer, N. I.; Glazman, I. M., Theory of linear operators in Hilbert space, vol. II, (1981), Pitman Boston · Zbl 0467.47001

[2] Amrein, W. O., Non-relativistic quantum dynamics, Mathematical Physics Studies, vol. 2, (1981), Reidel Dordrecht · Zbl 0466.47001

[3] Behncke, H., Spectral properties of the Dirac equation with anomalous magnetic moment, J. Math. Phys., 26, 2556-2559,

(1985) · Zbl 0578.47012

[4] Behncke, H.; Focke, H., Stability of deficiency indices, Proc. Roy. Soc. Edinburgh Sect. A, 78, 119-127, (1977) · Zbl 0398.47009

[5] Behncke, H.; Focke, H., Deficiency indices of singular Schrödinger operators, Math. Z., 158, 87-98, (1978) · Zbl 0353.35033

[6] Braverman, M.; Milatovich, O.; Shubin, M., Essential selfadjointness of Schrödinger-type operators on manifolds, Russian Math. Surveys, 57, 4, 641-692, (2002) · Zbl 1052.58027

[7] Brézis, H., "localized" self-adjointness of Schrödinger operators, J. Operator Theory, 1, 287-290, (1979) · Zbl 0439.35021

[8] Brusentsev, A. G., Selfadjointness of elliptic differential operators in $L_2(G)$, and correction potentials, Trans. Moscow Math. Soc., 31-61, (2004) · Zbl 1177.35058

[9] Bulla, W.; Gesztesy, F., Deficiency indices and singular boundary conditions in quantum mechanics, J. Math. Phys., 26, 2520-2528, (1985) · Zbl 0583.35029

[10] Cazacu, C., New estimates for the Hardy constant of multipolar Schrödinger operators · Zbl 1360.46026

[11] Combescure, M.; Ginibre, J., Scattering and local absorption for the Schrödinger operator, J. Funct. Anal., 29, 54-73, (1978) · Zbl 0382.47004

[12] Cuenin, J.-C.; Siedentop, H., Dipoles in graphene have infinitely many bound states, J. Math. Phys., 55, 122304, (2014) · Zbl 1311.82055

[13] Cycon, H. L., A theorem on "localized" self-adjointness of Schrödinger operators with $L_{\operatorname{loc}}^1$-potentials, Int. J. Appl. Math. Sci., 5, 545-552, (1982) · Zbl 0497.35025

[14] Cycon, H. L.; Froese, R. G.; Kirsch, W.; Simon, B., Schrödinger operators with applications to quantum mechanics and global geometry, Texts and Monographs in Physics, (1987), Springer Berlin · Zbl 0619.47005

[15] Devinatz, A., Essential self-adjointness of Schrödinger-type operators, J. Funct. Anal., 25, 58-69, (1977) · Zbl 0346.35040

[16] Eastham, M. S.P.; Evans, W. D.; McLeod, J. B., The essential self-adjointness of Schrödinger-type operators, Arch. Ration. Mech. Anal., 60, 185-204, (1976) · Zbl 0326.35018

[17] Edmunds, D. E.; Evans, W. D., Spectral theory and differential operators, (1989), Clarendon Press Oxford · Zbl 0664.47014

[18] Fall, M. M.; Felli, V., Sharp essential self-adjointness of relativistic Schrödinger operators with a singular potential, J. Funct. Anal., 267, 1851-1877, (2014) · Zbl 1295.35377

[19] Faris, W. G., Self-adjoint operators, Lecture Notes in Mathematics, vol. 433, (1975), Springer Berlin · Zbl 0317.47016

[20] Felli, V.; Marchini, E. M.; Terracini, S., On Schrödinger operators with multipolar inverse-square potentials, J. Funct. Anal., 250, 265-316, (2007) · Zbl 1222.35074

[21] Felli, V.; Marchini, E. M.; Terracini, S., On the behavior of solutions to Schrödinger equations with dipole type potentials near the singularity, Discrete Contin. Dyn. Syst., 21, 91-119, (2008) · Zbl 1141.35362

[22] Felli, V.; Marchini, E. M.; Terracini, S., On Schrödinger operators with multisingular inverse-square anisotropic potentials, Indiana Univ. Math. J., 58, 617-676, (2009) · Zbl 1169.35013

[23] Ferrero, P.; De Pazzis, O.; Robinson, D. W., Scattering theory with singular potentials. II. the \textit{N}-body problem and hard cores, Ann. Inst. Henri Poincaré, 21, 217-231, (1974)

[24] Frehse, J., Essential selfadjointness of singular elliptic operators, Bol. Soc. Bras. Mat., 8, 2, 87-107, (1977) · Zbl 0448.47029

[25] Gesztesy, F.; Weikard, R., Some remarks on the spectral problem underlying the Camassa-Holm hierarchy, (Ball, J. A.; Drietschel, M. A.; ter Elst, A. F.M.; Portal, P.; Potapov, D., Operator Theory in Harmonic and Non-commutative Analysis, IWOTA 12, Operator Theory: Advances and Applications, vol. 240, (2014), Birkhäuser, Springer Basel), 137-188 · Zbl 1322.47028

[26] Hunziker, W.; Günther, C., Bound states in dipole fields and continuity properties of electronic spectra, Helv. Phys. Acta, 53, 201-208, (1980)

[27] Ikebe, T.; Kato, T., Uniqueness of the self-adjoint extension of singular elliptic differential operators, Arch. Ration. Mech. Anal., 9, 77-92, (1962) · Zbl 0103.31801

[28] Ismagilov, R. S., Conditions for the semiboundedness and discreteness of the spectrum for one-dimensional differential equations, Sov. Math., Dokl., 2, 1137-1140, (1961) · Zbl 0286.34031

[29] Jörgens, K., Wesentliche selbstadjungiertheit singulärer elliptischer differentialoperatoren zweiter ordnung in $C_0^\infty(G)$, Math. Scand., 15, 5-17, (1964) · Zbl 0132.07601

[30] Jörgens, K., Perturbations of the Dirac operator, (Everitt, W. N.; Sleeman, B. D., Conference on the Theory of Ordinary and Partial Differential Equations, Lecture Notes in Mathematics, vol. 280, (1972), Springer Berlin) · Zbl 0245.35070

[31] Kalf, H., Self-adjointness for strongly singular potentials with a $- | x |^2$ fall-off at infinity, Math. Z., 133, 249-255, (1973) · Zbl 0266.35018

[32] Kalf, H., Gauss' theorem and the self-adjointness of Schrödinger operators, Ark. Mat., 18, 19-47, (1980) · Zbl 0458.35025

[33] Kalf, H., A note on the domain characterization of certain Schrödinger operators with strongly singular potentials, Proc. Roy. Soc. Edinburgh Sect. A, 97A, 125-130, (1984) · Zbl 0544.47041

[34] Kalf, H.; Rofe-Beketov, F. S., On the essential self-adjointness of Schrödinger operators with locally integrable potentials, Proc. Roy. Soc. Edinburgh Sect. A, 128A, 95-106, (1998) · Zbl 0892.35045

[35] Kalf, H.; Schmincke, U.-W.; Walter, J.; ust, R. W., On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials, (Everitt, W. N., Spectral Theory and Differential Equations, Lecture Notes in Math., vol. 448, (1975), Springer Berlin), 182-226 · Zbl 0311.47021

 

[36] Kalf, H.; Walter, J., Strongly singular potentials and essential self-adjointness of singular elliptic operators in $C_0^\infty(\mathbb{R}^n \backslash \{0 \})$, J. Funct. Anal., 10, 114-130, (1972) · Zbl 0229.35041

[37] Kalf, H.; Walter, J., Note on a paper of Simon on essentially self-adjoint Schrödinger operators with singular potentials, Arch. Ration. Mech. Anal., 52, 258-260, (1973) · Zbl 0277.47008

[38] Karlsson, B., Selfadjointness of Schrödinger operators, (1976), Inst. Mittag-Leffler, Report No. 6

[39] Karnarski, B., Matrix-differentialoperatoren erster ordnung und ihre anwendung auf die Dirac-gleichung mit anomalen magnetischem moment, (1982), University of Osnabrück, PhD thesis · Zbl 0527.47031

[40] Karnarski, B., Generalized Dirac-operators with several singularities, J. Operator Theory, 13, 171-188, (1985) · Zbl 0569.47044

[41] Kato, T., Schrödinger operators with singular potentials, Israel J. Math., 13, 135-148, (1972)

[42] Kato, T., A second look at the essential selfadjointness of the Schrödinger operators, (Enz, C. P.; Mehra, J., Physical Reality and Mathematical Description, (1974), Reidel Dordrecht), 193-201

[43] Kato, T., Perturbation theory for linear operators, (1980), Springer Berlin, corr. printing of the 2nd ed.

[44] Kato, T., Remarks on the selfadjointness and related problems for differential operators, (Knowles, I. W.; Lewis, R. T., Spectral Theory of Differential Operators, North-Holland Mathematics Studies, vol. 55, (1981), Elsevier, North-Holland Amsterdam) · Zbl 0484.35027

[45] Keller, R. G., The essential self-adjointness of differential operators, Proc. Roy. Soc. Edinburgh Sect. A, 82A, 305-344, (1979) · Zbl 0403.35030

[46] Kirsch, W., Über spektren stochastischer schrödingeroperatoren, (1981), Ruhr-Universität Bochum, PhD thesis · Zbl 0531.35066

[47] Kittel, C., Introduction to solid state physics, (2005), Wiley

[48] Klaus, M., Dirac operators with several Coulomb singularities, Helv. Phys. Acta, 53, 463-482, (1980)

[49] Klaus, M.; Wüst, R., Characterization and uniqueness of distinguished self-adjoint extensions of Dirac operators, Comm. Math. Phys., 64, 171-176, (1979) · Zbl 0408.47022

[50] Klaus, M.; Wüst, R., Spectral properties of Dirac operators with singular potentials, J. Math. Anal. Appl., 72, 206-214, (1979) · Zbl 0423.47014

[51] Knowles, I., On essential self-adjointness for singular elliptic differential operators, Math. Ann., 227, 155-172, (1977) · Zbl 0344.35026

[52] Knowles, I., On the existence of minimal operators for Schrödinger-type differential expressions, Math. Ann., 233, 221-227, (1978) · Zbl 0376.35016

[53] Kostenko, A.; Sakhnovich, A.; Teschl, G., Inverse eigenvalue problems for perturbed spherical Schrödinger operators, Inverse Probl., 26, (2010) · Zbl 1227.34023

[54] Ludwig, W., Festkörperphysik, (1978), Akademische Verlagsgesellschaft Wiesbaden, Studien - Text, Physik

[55] Maeda, M., Essential self-adjointness of Schrödinger operators with potentials singular along affine subspaces, Hiroshima Math. J., 11, 275-283, (1981) · Zbl 0513.35025

[56] Maz'ya, V. G.; Shaposhnikova, T. O., Theory of Sobolev multipliers. with applications to differential and integral operators, (2009), Springer Berlin · Zbl 1157.46001

[57] Maz'ya, V. G.; Verbitsky, I. E., The Schrödinger operator on the energy space: boundedness and compactness criteria, Acta Math., 188, 263-302, (2002) · Zbl 1013.35021

[58] Maz'ya, V. G.; Verbitsky, I. E., Infinitesimal form boundedness and Trudinger's subordination for the Schrödinger operator, Invent. Math., 162, 81-136, (2005) · Zbl 1132.35353

[59] Mitrea, D.; Mitrea, I.; Mitrea, M.; Monniaux, S., Groupoid metrization theory. with applications to analysis on quasi-metric spaces and functional analysis, Applied and Numerical Harmonic Analysis, (2013), Birkhäuser Basel · Zbl 1269.46002

[60] Morgan, J. D., Schrödinger operators whose potentials have separated singularities, J. Operator Theory, 1, 109-115, (1979) · Zbl 0439.35022

[61] Nenciu, G., Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms, Comm. Math. Phys., 48, 235-247, (1976) · Zbl 0349.47014

[62] Nenciu, G., Distinguished self-adjoint extension for Dirac operator with potential dominated by multicenter Coulomb potentials, Helv. Phys. Acta, 50, 1-3, (1977)

[63] Oleinik, I. M., On the essential self-adjointness of the general second order elliptic operators, Proc. Amer. Math. Soc., 127, 889-900, (1999) · Zbl 0937.35034

[64] Pearson, D. B., General theory of potential scattering with absorption at local singularities, Helv. Phys. Acta, 48, 639-653, (1975)

[65] Pearson, D. B., Quantum scattering and spectral theory, (1988), Academic Press San Diego · Zbl 0673.47011

[66] Persson, A., Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator, Math. Scand., 8, 143-153, (1960) · Zbl 0145.14901

[67] Reed, M.; Simon, B., Methods of modern mathematical physics. II: Fourier analysis, self-adjointness, (1975), Academic Press New York · Zbl 0308.47002

[68] Reed, M.; Simon, B., Methods of modern mathematical physics. IV: analysis of operators, (1978), Academic Press New York · Zbl 0401.47001

[69] Robinson, D. W., Scattering theory with singular potentials. I. the two-body problem, Ann. Inst. Henri Poincaré, 21, 185-215, (1974)

[70] Rozenblum, G. V.; Shubin, M. A.; Solomyak, M. Z., Spectral theory of differential operators, (Shubin, M. A., Partial Differential Equations VII, Ecyclopaedia of Mathematical Sciences, vol. 64, (1994), Springer Berlin)

[71] Schechter, M., Essential self-adjointness of the Schrödinger operator with magnetic vector potential, J. Funct. Anal., 20, 93-104, (1975) · Zbl 0323.35022

[72] Schechter, M., Operator methods in quantum mechanics, (1981), North Holland New York · Zbl 0456.47012

[73] Schechter, M., Spectra of partial differential operators, (1986), North-Holland, Elsevier Amsterdam · Zbl 0607.35005

[74] Schmincke, U.-W., Essential selfadjointness of a Schrödinger operator with strongly singular potential, Math. Z., 124, 47-50, (1972) · Zbl 0225.35037

[75] Schmüdgen, K., Unbounded self-adjoint operators on Hilbert space, Graduate Texts in Mathematics, vol. 265, (2012), Springer Dordrecht · Zbl 1257.47001

[76] Shubin, M., Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds, J. Funct. Anal., 186, 92-116, (2001) · Zbl 0997.58021

[77] Simader, C. G., Essential self-adjointness of Schrödinger operators bounded from below, Math. Z., 159, 47-50, (1978) · Zbl 0409.35026

[78] Simon, B., Essential self-adjointness of Schrödinger operators with positive potentials, Math. Ann., 201, 211-220, (1973) · Zbl 0234.47027

[79] Simon, B., Essential self-adjointness of Schrödinger operators with singular potentials, Arch. Ration. Mech. Anal., 52, 44-48, (1973) · Zbl 0277.47007

[80] Stetkær-Hansen, H., A generalization of a theorem of wienholtz concerning essential selfadjointness of singular elliptic operators, Math. Scand., 19, 108-112, (1966) · Zbl 0149.07502

[81] Stummel, F., Singuläre elliptische differentialoperatoren in hilbertschen Räumen, Math. Ann., 132, 150-176, (1956) · Zbl 0070.34603

[82] Teschl, G., Mathematical methods in quantum mechanics. with applications to Schrödinger operators, Graduate Studies in Mathematics, vol. 157, (2014), Amer. Math. Soc. Providence · Zbl 1342.81003

[83] von Wahl, W., A remark to a paper of Kato and ikebe, Manuscripta Math., 20, 197-208, (1977) · Zbl 0407.35029

[84] Walter, J., Symmetrie elliptischer differentialoperatoren, Math. Z., 98, 401-406, (1967) · Zbl 0146.34302

[85] Walter, J., Note on a paper by stetkær-hansen concerning essential selfadjointness of schroedinger operators, Math. Scand., 25, 94-96, (1969) · Zbl 0184.32702

[86] Weidmann, J., Linear operators in Hilbert spaces, Graduate Texts in Mathematics, vol. 68, (1980), Springer New York

[87] Wienholtz, E., Halbbeschränkte partielle differentialoperatoren zweiter ordnung vom elliptischen typus, Math. Ann., 135, 50-80, (1958) · Zbl 0142.37701