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Infinite-dimensional dynamical systems in mechanics and physics. (English) Zbl 0662.35001

Applied Mathematical Sciences, 68. New York etc.: Springer-Verlag. xvi, 500 p. DM 124.00 (1988).

This book is devoted to the onset of turbulence in infinite-dimensional dynamical systems or flows. The author himself has made a number of important contributions to this field. These systems are usually derived from partial differential equations of evolution, associated with boundary value problems. Let us say that $du/dt = F_\lambda(u)$ is the equation of evolution with an initial condition $u(0) = u_0$. For any $t \geq 0$, u(t) is an element of some Hilbert or Banach space H, and we are interested mainly in the behavior of the solution u(t) as $t \to +\infty$. In contrast with the ordinary differential equations case, there are no general theorems of existence and uniqueness of solutions. Each case has to be considered separately, with different function spaces being involved. The author combines the functional analysis of equations of evolution with the theory of dynamical systems to study chaotic (turbulent) behavior of some relevant specific equations of mathematical physics. These include reaction-diffusion equations, Navier-Stokes in space dimension 2, flows on a manifold (for geophysics), thermohydraulics, sine-Gordon, damped nonlinear wave equations, etc. A "user's guide" for physics-oriented readers who are not interested in the technical aspects is included in the preface.

The λ in the above nonlinear operator F_λ is a bifurcation parameter, like a Reynolds number or a Grashof number, for example. The behavior of the solutions changes dramatically as we vary this parameter. For some values of λ we may have that many solutions are asymptotic to stationary solutions, to periodic orbits or to tori. For some other values of λ, we get irregular, apparently chaotic solutions which are called turbulent. Following ideas of *D. Ruelle* and *F. Takens* [Commun. Math. Phys. 20, 167-192 (1971; Zbl 0223.76041), ibid. 23, 343-344 (1971; Zbl 0227.76084)], this is because the system becomes dissipative and solutions wander close to a complicated attractor $\mathcal{A}$, typically a Cantor set or the product of a Cantor set with an interval. This means roughly that any given solution has as ω-limit set (i.e., the limit points as t goes to $+\infty$) a compact set $K \subset H$, which is part of the "large" compact invariant set $\mathcal{A}$, attracting all the orbits.

Turbulence is essentially a finite dimensional phenomenon, since it may appear for dynamical systems in dimension 3, e.g. the classical Lorenz attractor. In infinite dimensions it may have more complicated aspects, but one of the relevant points in the book is the claim that the number of degrees of freedom of dissipative phenomena when the "permanent regime" is established, is finite (but very large). This number N is understood as the Hausdorff dimension of the attractor $\mathcal{A}$. By estimates on the Lyapunov exponents, actual explicit upper bounds are given on N and on the fractal dimension of $\mathcal{A}$ for some physical systems, getting figures of the order of 10^9 or 10^{20} (Chapter VI). If m is any integer upper bound for N, a result by *R. Mañé* [Lect. Notes Math. 898, 230-242 (1981; Zbl 0544.58014)] asserts the flow on $\mathcal{A}$ can be parametrized by the Euclidean space of dimension $2m + 1$. Choosing the relevant parameters and being able to handle simulations by computers are still open problems of current research.

For a general class of problems of fluid dynamics, well-posedness is shown to imply dissipativity of the system (Chapter VII). For the Navier- Stokes equation in space dimension 3, turbulence is more complex due to the appearance of singularities and non-well-posedness. However, by using his own techniques, the author is able to give also an upper bound on the dimension of the attractor for this case. On the other hand, in general a lower bound on the (Hausdorff or fractal dimension) of $\mathcal{A}$ is given as the dimension of the unstable manifold of any suitable hyperbolic fixed point in $\mathcal{A}$.

We end up by mentioning that in the last chapter of the book inertial manifolds are defined. They are finite dimensional Lipschitz manifolds attracting all the orbits and containing the attractor, which gives a more convenient reduction of the system to finite dimension. This is a promising new object which enjoys stability with respect to certain perturbations, and much current research goes on about it.

Reviewer: E.A.Lacomba

MSC:

35-02	Research exposition (monographs, survey articles) pertaining to partial differential equations
35B40	Asymptotic behavior of solutions to PDEs
35G10	Initial value problems for linear higher-order PDEs
35K25	Higher-order parabolic equations
37D45	Strange attractors, chaotic dynamics of systems with hyperbolic behavior
35B32	Bifurcations in context of PDEs
35Q30	Navier-Stokes equations
35K57	Reaction-diffusion equations
35A05	General existence and uniqueness theorems (PDE) (MSC2000)
35L70	Second-order nonlinear hyperbolic equations
35Q99	Partial differential equations of mathematical physics and other areas of application

Keywords:

turbulence; infinite-dimensional dynamical systems; flows; equations of evolution; existence; uniqueness; function spaces; chaotic; reaction- diffusion; Navier-Stokes; flows on a manifold; sine-Gordon; damped nonlinear wave equations; bifurcation; periodic orbits; tori; turbulent; dissipative; complicated attractor; Lorenz attractor; fractal dimension; well-posedness; inertial manifolds