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**Bayes Hilbert spaces.** (English) Zbl 1335.62025
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Summary: A Bayes linear space is a linear space of equivalence classes of proportional $\sigma$-finite measures, including probability measures. Measures are identified with their density functions. Addition is given by Bayes' rule and substraction by Radon-Nikodym derivatives. The present contribution shows the subspace of square-log-integrable densities to be a Hilbert space, which can include probability and infinite measures, measures on the whole real line or discrete measures. It extends the ideas from the Hilbert space of densities on a finite support towards Hilbert spaces on general measure spaces. It is also a generalisation of the Euclidean structure of the simplex, the sample space of random compositions. In this framework, basic notions of mathematical statistics get a simple algebraic interpretation. A key tool is the centred-log-ratio transformation, a generalization of that used in compositional data analysis, which maps the Hilbert space of measures into a subspace of square-integrable functions. As a consequence of this structure, distances between densities, orthonormal bases, and Fourier series representing measures become available. As an application, Fourier series of normal distributions and distances between them are derived, and an example related to grain size distributions is presented. The geometry of the sample space of random compositions, known as Aitchison geometry of the simplex, is obtained as a particular case of the Hilbert space when the measures have discrete and finite support.

### MSC:

62C10 Bayesian problems; characterization of Bayes procedures
60B11 Probability theory on linear topological spaces



### Keywords:

Aitchison geometry of the simplex; distance between measures; Fourier coefficients; infinite measures; normal distribution; perturbation; probability measures

### Software:

sm

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