Cutolo, Giovanni; Smith, Howard

Groups with countably many subgroups. (English) Zbl 1342.20030

J. Algebra 448, 399-417 (2016).

It is an easy observation that a group is finite precisely when it has only finitely many subgroups. In the very nice paper under review the authors consider groups G in which the set of all subgroups of G is countable. The abelian groups with this property are classified in [*S. V. Rychkov* and *A. A. Fomin*, "Abelian groups with a countable number of subgroups", Abelevy Gruppy Moduli 10, 99-105 (1991)]. The current paper is mostly concerned with soluble such groups. A group G is termed a CMS-group if $\mathcal{L}(G)$, the set of all subgroups of G is countable. Of course, all CMS-groups are countable and also groups with the maximum condition are CMS-groups. The class of CMS-groups is closed under taking subgroups and quotients, but $C_{p^\infty} \times C_{p^\infty}$ (which has $2^{\aleph_0}$ subgroups) shows that the property is not extension closed.

The authors show (Theorem 2.7) that a soluble-by-finite group is CMS if and only it is minimax and has no (subnormal) sections of type $C_{p^\infty} \times C_{p^\infty}$. They also show that locally (soluble-by-finite) CMS-groups are soluble-by-finite (Theorem 2.12). Interestingly also (Theorem 3.1) they give an example of a nilpotent group with uncountably many subgroups, of which just countably many are abelian. On the other hand, if G is an infinite soluble-by-finite group then $|\mathcal{L}(G)|$ is either $\aleph_0$ or $2^{|G|}$. The construction of Theorem 35.2 of *A. Yu. Ol'shanskij* [Geometry of defining relations in groups. Dordrecht: Kluwer Academic Publishers (1991; Zbl 0732.20019)] provides a simple group of cardinality $\aleph_1$ with exactly $\aleph_1$ subgroups, all proper subgroups are countable and G may be arranged to be a p-group, for a sufficiently large prime p.

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MSC:

20E15 Chains and lattices of subgroups, subnormal subgroups
20E07 Subgroup theorems; subgroup growth
20F16 Solvable groups, supersolvable groups
20F19 Generalizations of solvable and nilpotent groups
20F50 Periodic groups; locally finite groups

Keywords:

soluble groups; CMS-groups; countable sets of subgroups; Abelian subgroups; soluble-by-finite groups; minimax groups; group with countably many subgroups

Full Text: DOI

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