Ernvall-Hytönen, Anne-Maria; Odžak, Almasa; Smajlović, Lejla; Sušić, Medina
On the modified Li criterion for a certain class of L-functions. (English) Zbl 1347.11063
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The authors consider a class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ of L-functions that contains the Selberg class, the class of all automorphic L-functions and the Rankin-Selberg L-functions, as well as products of suitable shifts of those functions. They prove the generalized Li criterion for zero-free regions of functions belonging to the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$, derive an arithmetic formula for τ-Li coefficients and conduct numerical investigation of τ-Li coefficients for a certain product of shifts of the Riemann zeta function.

More precisely, for real numbers σ_0 and σ_1 such that $\sigma_0 \geq \sigma_1 > 0$, the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ is the class of functions F satisfying the following four axioms:

- (Dirichlet series) The function F possesses a Dirichlet series representation that converges absolutely for $\mathrm{Re}(s) > \sigma_0$.
- (Analytic continuation) There exist finitely many non-negative integers $m_1,\dots,m_N$ and complex numbers $s_1,\dots,s_N$ such that the function $\prod_{i=1}^{N}(s-s_i)^{m_i}F(s)$ is an entire function of finite order.
- (Functional equation) The function F satisfies the functional equation $\xi_F(s)=\omega\overline{\xi_F(\sigma_1-\bar{s})}$, where the completed function ξ_F is defined as

$$\begin{aligned}\xi_F(s) = & F(s)Q_F^s\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)\prod_{i=1}^{2M+\delta(\sigma_1)}(s-s_i)^{m_i}\\ & \prod_{i=2M+1+\delta(\sigma_1)}^{N}(s-s_i)^{m_i}(\sigma_1-s-\overline{s_i})^{m_i},\end{aligned}$$

  where $|\omega|=1$, $Q_F>0$, $r\geq 0$, $\lambda_j>0$, $\mu_j\in\mathbb{C}$, $j=1,\dots,r$, and Γ is the Euler Gamma function. It is assumed that the poles of F are arranged so that the first $2M+\delta(\sigma_1)$ poles $(0\leq 2M+\delta(\sigma_1)\leq N)$ are such that $s_{2j-1}+\overline{s}_{2j}=\sigma_1$, for $j=1,\dots,M$, and $\delta(\sigma_1)=1$ if $\sigma_1/2$ is a pole of F in which case $s_{2M+\delta(\sigma_1)}=\sigma_1/2$; otherwise $\delta(\sigma_1)=0$.
- (Euler sum) The logarithmic derivative of the function F possesses a Dirichlet series representation

converging absolutely for $\mathrm{Re}(s)>\sigma_0$.

The *non-trivial zeros* of F are defined to be the zeros of the completed function ξ_F. The set of non-trivial zeros of $F(s)$ is denoted by $Z(F)$. By the functional equation and the Euler sum, all those zeros lie in the critical strip $\sigma_1-\sigma_0\leq\mathrm{Re}(s)\leq\sigma_0$. Let $\tau\in[\sigma_1,+\infty)$. For an arbitrary positive integer n, the nth τ-Li coefficient associated to the $F\in\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ is defined as

$$\lambda_F(n,\tau)=\sum_{\rho\in Z(F)}\left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right),$$

where the sum is taken in the sense of the limit $\lim\limits_{T\to\infty}\sum\limits_{|\mathrm{Im}(\rho)|\leq T}$. The main result of the paper is the following Li-type criterion. Let $0,\tau\notin Z(F)$. The next two statements are equivalent

(i) $\sigma_1-\frac{\tau}{2}\leq\mathrm{Re}(\rho)\leq\frac{\tau}{2}$ for every $\rho\in Z(F)$,

(ii) $\mathrm{Re}(\lambda_F(n,\tau))\geq 0$ for every positive integer n.

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MSC:

11M26	Nonreal zeros of $\zeta(s)$ and $L(s,\chi)$; Riemann and other hypotheses	Cited in **4** Documents
11M36	Selberg zeta functions and regularized determinants; applications to spectral theory, Dirichlet series, Eisenstein series, etc. (explicit formulas)	
11M41	Other Dirichlet series and zeta functions	

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L-functions; modified Li coefficients; generalized Li criterion

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