

Hindes, Wade

The arithmetic of curves defined by iteration. (English) Zbl 1330.14032
Acta Arith. 169, No. 1, 1-27 (2015).

Given an irreducible quadratic polynomial $f(x) = f_c(x) = x^2 + c \in \mathbb{Q}[x]$ whose n -th iterate f^n has distinct roots, let \mathbb{T}_n be $\{0\} \coprod f^{-1}(0) \coprod \dots \coprod f^{-n}(0)$. This becomes a 2-ary rooted tree when we draw an edge between α and β whenever $f(\alpha) = \beta$. The Galois group G_n of f^n acts on \mathbb{T}_n , and we can ask for which c finiteness of $[\text{Aut}(\mathbb{T}_n) : G_n]$ holds (and in the limit $n = \infty$). This is a dynamical analog of Serre's open image conjecture.

This paper focuses on two aspects of this problem. First, in Theorem 1.1, he studies c 's for which $n = 4$ is the first non-maximality, i.e. $G_3 = \text{Aut}(\mathbb{T}_3)$ but $G_4 \neq \text{Aut}(\mathbb{T}_4)$. In particular, no such c exists for $c \in \mathbb{Z}$, and only such $c \in \mathbb{Q}$ is $\frac{2}{3}$ and $-\frac{6}{7}$ as long as a certain curve has no rational points above a certain height. Secondly, the author shows in Theorem 1.2 that the Hall–Lang conjecture implies finiteness of $[\text{Aut}(\mathbb{T}_\infty) : G_\infty]$ for integers c which are not negatives of squares, and shows that this index is 2 when $c = 3$.

To prove these results, the author considers curves $C_{c,n} : y^2 = f_c^n(x)$ and $B_{c,n} : y^2 = (x - c)f_c^n(x)$, as well as their twists. By using *M. Stoll's* criterion [*Arch. Math.* 59, No. 3, 239–244 (1992; [Zbl 0758.11045](#))], the author relates the non-maximality to rational points on certain curves. More specifically, for Theorem 1.2, he constructs rational points on twists of $C_{c,n}$ and $B_{c,1}$. For Theorem 1.1, he shows that $\sqrt{f_c^4(0)}$ must be fixed by one of the 7 distinct index-2 subgroups of G_3 if $n = 4$ is the first non-maximality, resulting in a rational point on the corresponding hyperelliptic curves. Then standard techniques such as Chabauty and Runge's method are used to find rational points.

In addition to these results, the author provides a detailed analysis of $B_{-2,n}$ and their Jacobians. In this Chebyshev case, he constructs characteristic polynomial of Frobenius for primes $\equiv \pm 3 \pmod{8}$ and determines $B_{-2,n}(\mathbb{Q})$. This leads to the decomposition of $J(C_{c,n})$ into simple factors when $f_c \equiv x^2 - 2$ modulo such primes.

Reviewer: Yu Yasufuku (Tokyo)

MSC:

- [14G05](#) Rational points
- [37P05](#) Arithmetic and non-Archimedean dynamical systems involving polynomial and rational maps
- [12F10](#) Separable extensions, Galois theory
- [37P15](#) Dynamical systems over global ground fields
- [11G30](#) Curves of arbitrary genus or genus $\neq 1$ over global fields
- [11G05](#) Elliptic curves over global fields
- [14H45](#) Special algebraic curves and curves of low genus
- [20E08](#) Groups acting on trees
- [37P55](#) Arithmetic dynamics on general algebraic varieties
- [14H25](#) Arithmetic ground fields for curves

Cited in 1 Review
Cited in 4 Documents

Keywords:

quadratic polynomial; dynamical arboreal representation; Hall–Lang conjecture; rational points on curves

Software:

Magma; SageMath

Full Text: [DOI](#) [arXiv](#)

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