Kapovich, Ilya; **Lustig, Martin**
Cannon-Thurston fibers for iwip automorphisms of F_N. (English) Zbl 1325.20035
J. Lond. Math. Soc., II. Ser. 91, No. 1, 203-224 (2015).

According to the notion of a Cannon-Thurston map studied by *J. W. Cannon* and *W. P. Thurston*, [Geom. Topol. 11, 1315-1355 (2007; Zbl 1136.57009)], in group-theoretic terms an analogous notion is developed by *M. Mitra* [in Geom. Funct. Anal. 7, No. 2, 379-402 (1997; Zbl 0880.57001); Topology 37, No. 3, 527-538 (1998; Zbl 0907.20038); Geom. Topol. Monogr. 1, 341-364 (1998; Zbl 0914.20034)].

If G is a word-hyperbolic group and H a word-hyperbolic subgroup, and if the inclusion $\iota\colon H \to G$ extends to a continuous map $\widehat{\iota}\colon \partial H \to \partial G$, then the map $\widehat{\iota}$ is called the Cannon-Thurston map. In particular if the Cannon-Thurston map exists, then it is unique. It is well known that if $H \leq G$ is a quasiconvex subgroup of a word-hyperbolic group G, then H is word-hyperbolic and the inclusion extends to a continuous topological embedding $\partial H \to \partial G$. Thus in this case the Cannon-Thurston map exists and, moreover, is injective. A result of Mitra [in loc. cit., Zbl 0907.20038] states that whenever $1 \to H \to G \to Q \to 1$ is a short exact sequence of word-hyperbolic groups, then the inclusion $H \leq G$ extends to a continuous Cannon-Thurston map $\widehat{\iota}\colon \partial H \to \partial G$. Until recently, it has been unknown whether there are any inclusions $H \leq G$ (with H and G word-hyperbolic) where the Cannon-Thurston map does not exist. In [Forum Math. Sigma 1, Article ID e3 (2013; Zbl 1276.20054)], *O. Baker* and *T. R. Riley* construct the first example of such an inclusion where the Cannon-Thurston map does not exist.

Let F_N be the free group of rank $N \geq 2$ and $\Phi \in \mathrm{Aut}(F_N)$, then the mapping torus group of Φ is $G_\Phi = F_N \rtimes_\Phi \langle t \rangle$. Since the inclusion $F_N \leq G_\Phi$ depends only on the outer automorphism class $\varphi \in \mathrm{Out}(F_N)$ of Φ, we have the short exact sequence $1 \to F_N \to G_\varphi \to \langle t \rangle \to 1$. If the group G_φ is word-hyperbolic (in this case the automorphism Φ (or φ) is called hyperbolic), then by the above mentioned Mitra's result, there does exist a continuous F_N-equivariant surjective Cannon-Thurston map $\widehat{\iota}\colon \partial F_N \to \partial G_\varphi$.

In the present paper the authors study this Cannon-Thurston map $\widehat{\iota}$. Before stating their main results we quote some definitions and terminology referring for details to the paper.

An automorphism $\Phi \in \mathrm{Aut}(F_N)$ or its associated outer automorphism $\varphi \in \mathrm{Out}(F_N)$ is called fully irreducible or 'iwip' if there is no non-trivial proper free factor of F_N which is mapped by any positive power of Φ to a conjugate of itself. – An automorphism $\Phi \in \mathrm{Aut}(F_N)$ or its associated outer automorphism $\varphi \in \mathrm{Out}(F_N)$ is called atoroidal if no positive power of Φ fixes any non-trivial conjugacy class $[w] \subseteq F_N$.

For any iwip automorphism $\varphi \in \mathrm{Out}(F_N)$ the following are equivalent.

(1) The automorphism φ is atoroidal.

(2) The automorphism φ is not induced by a homeomorphism of a surface with boundary.

(3) The mapping torus group G_φ is word-hyperbolic.

A point $S \in \partial G_\varphi$ is called rational if it is the fixed point of an element $g \in G_\varphi \setminus \{1\}$. If $S = \lim_{n\to\infty} g^n$ (in the topology of the Gromov compactification of hyperbolic groups), then we write $S = g^\infty$.

Let $S \in \partial G_\varphi$. The degree $\deg(S)$ of S denotes the cardinality of the full preimage of S under the map $\widehat{\iota}\colon \partial F_N \to \partial G_\varphi$.

The following classes of points $S \in \partial G_\varphi$ are defined: The point S is simple if $\deg(S) = 1$. The point S is regular if $\deg(S) = 2$. The point S is singular if $\deg(S) \geq 3$. – The regular and singular points are subdivided into two types. The point S is of φ-type if for every two distinct $\widehat{\iota}$-preimages $X, Y \in \partial F_N$ of S, $(X, Y) \in L(T_-)$. The point S is of φ^{-1}-type if for every two distinct $\widehat{\iota}$-preimages $X, Y \in \partial F_N$ of S, $(X, Y) \in L(T_+)$; where $L(T_-)$ and $L(T_+)$ are laminations defined in the paper.

Theorem 1. Let $\varphi \in \mathrm{Out}(F_N)$ be an atoroidal iwip and let $\widehat{\iota}\colon \partial F_N \to \partial G_\varphi$ be the Cannon-Thurston map. Then one has $\sum(\deg([S]_{F_N}) - 2) \leq 2N - 2$, where the summation is taken over all F_N-orbits $[S]_{F_N}$ of singular points $S \in \partial G_\varphi$ that are of φ-type.

The same inequality holds if the summation is taken over all F_N-orbits $[S]_{F_N}$ of singular points of φ^{-1}-type.

Theorem 2. Let $\varphi \in \mathrm{Out}(F_N)$ be an atoroidal iwip and let $\widehat{\imath}\colon \partial F_N \to \partial G_\varphi$ be the Cannon-Thurston map. Then the following hold.

(1) For every $S \in \partial G_\varphi$, we have $\deg(S) \leq 2N$.

(2) The number of F_N-orbits of singular points of φ-type (respectively, of φ^{-1}-type) in ∂G_φ satisfies $\mathrm{card}\{F_N \cdot S \subseteq \partial G_\varphi \mid S \text{ singular of } \varphi\text{-type }\} \leq 2N-2$.

(3) Every singular point $S \in \partial G_\varphi$ is rational. More precisely, there exists $g \in G_\varphi \setminus F_N$ such that $S = g^\infty$.

Theorem 3. Let $\varphi \in \mathrm{Out}(F_N)$ be an atoroidal iwip, let $\widehat{\imath}\colon \partial F_N \to \partial G_\varphi$ be the Cannon-Thurston map and let $g \in G_\varphi \setminus \{1\}$ be arbitrary. Then $\deg(g^\infty) + \deg(g^{-\infty}) \leq 4N-1$.

The upper bounds given in the theorems above are sharp. (For a concrete example, for every $N \geq 3$ the authors refer to [*A. Jäger* and *M. Lustig*, Geom. Topol. Monogr. 14, 321-333 (2008; Zbl 1140.20027)].)

The paper concludes with the use of a proposition (Proposition 4.5 in the paper) to fill a gap in the proof of a Theorem of *M. Mitra* [in Proc. Am. Math. Soc. 127, No. 6, 1625-1631 (1999; Zbl 0918.20028)] (a correction obtained already by Mitra himself [in "On a theorem of Scott and Swarup", arXiv:1209.4165]).

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MSC:

20F65	Geometric group theory
20F67	Hyperbolic groups and nonpositively curved groups
20E36	Automorphisms of infinite groups
20E05	Free nonabelian groups
57M07	Topological methods in group theory
37B10	Symbolic dynamics
57M50	General geometric structures on low-dimensional manifolds

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word-hyperbolic groups; Cannon-Thurston map; iwip automorphisms; fully irreducible automorphisms; mapping torus groups

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