Kurke, Herbert; **Osipov, Denis**; **Zheglov, Alexander**
Commuting differential operators and higher-dimensional algebraic varieties. (English)
Zbl 1306.37077
Sel. Math., New Ser. 20, No. 4, 1159-1195 (2014).

The authors study algebro-geometric properties of commutative rings of partial differential operators (PDOs). In the theory of algebraic integrable systems, it is interesting to find explicit examples of certain commutative rings of PDOs. The case of $n = 1$ variable concerns the method of constructing explicit solutions for various nonlinear integrable equations, in particular the Korteweg-de Vries (KdV) and the Kadomtsev-Petviashvili (KP) equations. One relevant question here is how can one find a ring of commuting ordinary differential operators that contains a pair of monic operators P, Q such that $\mathbb{C}[P, Q] \not\cong \mathbb{C}[u]$. The classification of such rings was obtained by purely algebraic methods and is given in the general case by Krichever, where the connection of the classification with integrable systems with the spectral operator theory and with the theory of linear differential equations with periodical coefficients is given. Simultaneously, a theory was developed in this field in connection with famous equations (KP, KdV, sine-Gordon, Toda).

For operators in $n > 1$ variables, the problem is to find a ring of commuting partial differential operators that concern $n + 1$ operators $L_0, \dots, L_n$ with algebraically independent homogeneous constant highest symbols $\sigma_1, \dots, \sigma_n$ such that $\mathbb{C}[\mathbb{C}^n]$ is finitely generated as a module over the ring generated by these σ's and L_0 is not a polynomial combination of $L_1, \dots, L_n$.

Certain conditions from the $n = 1$ case can be generalized to such rings, e.g., the analogue of the Burchnall-Chaundy lemma saying that $n + 1$ commuting operators $L_0, \dots, L_n$ are algebraically dependent. It was shown that for a ring of commuting partial differential operators satisfying certain properties, there is a unique Baker-Akhieser function that completely characterizes the ring by its spectral variety.

There are only few examples known of the rings for $n > 1$ and these are all connected with the quantum (deformed) Calogero-Moser systems. Also, there is a construction of a free BA-module developed to produce explicit examples of commuting matrix rings of PDO's.

The result about commutative rings of PDO's says that there is a construction that associates to such a ring of commuting operators some algebro-geometric data that consist of a complete (projective) affine spectral variety, the divisor at infinity, a torsion-free sheaf of rank one, and some extra trivialization data. This is an analogue of the construction coming from the $n = 1$ case.

It is not known which geometric data exactly describe the commutative rings of PDOs, but these describe commutative rings of completed PDOs in the $n = 2$ case. Also, there is the approach considering a wider class of operators, the operators from the complete ring $\hat{D}$ of differential operators. All commutative subrings of $\hat{D}$ satisfying certain mild conditions are classified in terms of Parshin's modified geometric data. Such rings contain all subrings of partial differential operators in two variables after a change of coordinates.

The authors explain the history connected with the above approach to some detail: The commutative rings of ODOs are classified in terms of geometric data, whose main geometric object is a projective curve. Classically, in the KP theory, there is a map that associates to each such data a pair of subspaces (A, W) called Schur pairs in the space $V = k((z))$, where $A \not\supseteq k$ is a stabilizer k-subalgebra of W in V. That is, $AW \subset W$ and W is a point of the infinite-dimensional Sato Grassmannian. Usually, this map is called the Krichever map. Parshin introduced an analogue of this map which associates to each geometric data a pair of subspaces $(\mathbb{A}, \mathbb{W})$ in the two-dimensional local field associated with the algebra $k((u))((t))$. This map is proved to be bijective.

To extend the Krichever-Parshin map and to prove that it is bijective, the authors introduce new geometric objects. These are called formal punctured ribbons and they come equipped with torsion-free coherent sheaves. Then, the bijection between the set of geometric data and the set of pairs of subspaces $(\mathbb{A}, \mathbb{W})$ (generalized Schur pairs) can be proved. On the other hand, Parshin considered a multi-variable analogue of the KP-hierarchy which when modified is related to algebraic surfaces and torsion-free sheaves on such surfaces, and to a wider class of geometric data consisting of ribbons and torsion-free sheaves on them.

This leads to the need of a description of the geometric structure of the Picard scheme of a ribbon. The scheme has a group structure and is an analogue of the Jacobian of a curve in the context of the classical KP theory. KP flows are defined on such schemes.

To classify commutative rings in $\hat{D}$, the Parshin geometric data were modified: The surface need not be Cohen-Macaulay, the ample divisor need not be Cartier, and the sheaf need not be a vector bundle. Then, the classification is also established in terms of modified Schur pairs which are pairs of subspaces (A, W) in $k[[u]]/((t))$ satisfying properties similar to the Schur pairs $(\mathbb{A}, \mathbb{W})$.

The main goal of this article is to give answers toward the following questions (citing directly from the authors' introduction):

1. Can the construction that associates to each Parshin's data the data with ribbon be extended to the set of modified Parshin's data?
2. If yes, what happens if we apply the combinatorial construction that reconstructs the Parshin data to ribbon data from its ribbon's data coming from modified Parshin's data?
3. What is the relationship between the Schur pairs $(\mathbb{A}, \mathbb{W})$ and modified Schur pairs (A, W)?

   The article goes through the definition and basic results in a more or less self-contained way.

Of course, this is toward the end of a long fairy tale and so some knowledge of the beginning is necessary. This is a great article to study as an introduction to the field of integrable systems and as an overview of the most applicable methods.

Reviewer: Arvid Siqveland (Kongsberg)

MSC:

37K10 Completely integrable infinite-dimensional Hamiltonian and Lagrangian systems, integration methods, integrability tests, integrable hierarchies (KdV, KP, Toda, etc.)
16S32 Rings of differential operators (associative algebraic aspects)
14J60 Vector bundles on surfaces and higher-dimensional varieties, and their moduli
37K20 Relations of infinite-dimensional Hamiltonian and Lagrangian dynamical systems with algebraic geometry, complex analysis, and special functions

Keywords:

commuting partial differential operators; algebraically integrable systems; Sato theory; algebraic KP theory; algebraic surfaces; two-dimensional local fields; Parshin theory; generalized Schur pairs

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