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Hyperbolicity of the complex of free factors. (English) Zbl 1348.20028

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From the introduction: We develop the geometry of folding paths in Outer space and, as an application, prove that the complex of free factors of a free group of finite rank is hyperbolic.

The complex of free factors of a free group $\mathbb{F}$ of rank n is the simplicial complex $\mathcal{F}$ whose vertices are conjugacy classes of proper free factors A of $\mathbb{F}$, and simplices are determined by chains $A_1 < A_2 < \cdots < A_k$....There is a very useful analogy between $\mathcal{F}$ and the curve complex $\mathcal{C}$ associated with a compact surface (with punctures) Σ. The vertices of $\mathcal{C}$ are isotopy classes of essential simple closed curves in Σ, and simplices are determined by pairwise disjoint curves.

More recently, the curve complex has been used in the study of the geometry of mapping class groups and of ends of hyperbolic 3-manifolds. The fundamental result on which this work is based is the theorem of *H. A. Masur* and *Y. N. Minsky* [Invent. Math. 138, No. 1, 103-149 (1999; Zbl 0941.32012)] that the curve complex is hyperbolic. In the low complexity cases when $\mathcal{C}$ is a discrete set one modifies the definition of $\mathcal{C}$ by adding an edge when the two curves intersect minimally. In the same way, we modify the definition of $\mathcal{F}$ when the rank is $n = 2$ by adding an edge when the two free factors (necessarily of rank 1) are determined by a basis of $\mathbb{F}$, i.e. whenever $\mathbb{F} = \langle a, b\rangle$, then $\langle a\rangle$ and $\langle b\rangle$ span an edge. In this way $\mathcal{F}$ becomes the standard Farey graph. The main result in this paper is:

Main Theorem. The complex $\mathcal{F}$ of free factors is hyperbolic.

The statement simply means that when the 1-skeleton of $\mathcal{F}$ is equipped with the path metric in which every edge has length 1, the resulting graph is hyperbolic.

MSC:

20E05 Free nonabelian groups
20F65 Geometric group theory
57M50 General geometric structures on low-dimensional manifolds

Keywords:

Outer space; complexes of free factors; free groups; hyperbolicity

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