Gratton, C.; **Nguyen, K.**; **Tucker, T. J.**

ABC **implies primitive prime divisors in arithmetic dynamics.** (English) Zbl 1291.37121

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This article proves the existence of primitive prime divisors in orbits of rational functions. Over a number field K, the proof assumes the *abc* conjecture, while the proof is unconditional when K is a characteristic 0 function field of transcendence degree 1. The result does not assume that 0 is a preperiodic point or a ramification point, unlike many of the previous results.

More specifically, let φ be a rational function of degree $d > 1$, and denote its n-th iterate by φ^n. We say that a prime $\mathfrak{p}$ of K is a primitive prime of $\varphi^n(\alpha) - \beta$ if $v_{\mathfrak{p}}(\varphi^n(\alpha) - \beta) > 0$ and $v_{\mathfrak{p}}(\varphi^m(\alpha) - \beta) \leq 0$ for all $m < n$. We say that $\mathfrak{p}$ is a square-free primitive prime if further $v_{\mathfrak{p}}(\varphi^n(\alpha) - \beta) = 1$. Theorem 1.1 shows that there is a primitive prime of $\varphi^n(\alpha) - \beta$ for all sufficiently large n if the following all hold: (1) α is not preperiodic, (2) φ is non-isotrivial if K is a function field, (3) β is not in the orbit of α, (4) $(\varphi^2)^{-1}(\beta) \neq \{\beta\}$. Furthermore, Theorem 1.2 shows that $\varphi^n(\alpha) - \beta$ has a square-free primitive prime for all sufficiently large n if (4) above is replaced by the stronger (4'): there exists an infinite sequence $\{\beta_n\}$ of noncritical points such that $\cdots \mapsto \beta_n \mapsto \cdots \mapsto \beta_1 \mapsto \beta$.

The key of the proof is to show that given $\epsilon > 0$ and a polynomial F without multiple roots, there exists a constant C such that

$$\sum_{\mathfrak{p}: v_{\mathfrak{p}}(F(z))>0} \lambda_{\mathfrak{p}}^{(1)}(F(z)) \geq (\deg F - 2 - \epsilon)h(z) + C$$

for all $z \in K$, where $\lambda_{\mathfrak{p}}^{(1)}$ is a local height (with respect to 0) truncated at 1. Over number fields (Proposition 3.4), this is based on [*A. Granville*, Int. Math. Res. Not. 1998, No. 19, 991–1009 (1998; Zbl 0924.11018)], and uses the *abc* conjecture as well as the Belyi map. Over function fields (Proposition 4.2), because of the absence of Belyi maps, the authors instead use Vojta's $1 + \epsilon$ conjecture, proved by *K. Yamanoi* [Acta Math. 192, No. 2, 225–294 (2004; Zbl 1203.30035)].

The dynamical input comes in Proposition 5.1, which shows that the sum of $\lambda_{\mathfrak{p}}^{(1)}(\varphi^n(\alpha))$ over good-reduction primes which divide $\varphi^m(\alpha)$ for some $m < n$ grow only $o(h(\varphi^n(\alpha)))$. By choosing F to be a suitable factor of some iterate of φ, the main results follow.

Building on [*M. Stoll*, Arch. Math. 59, No. 3, 239–244 (1992; Zbl 0758.11045)], the authors apply Theorem 1.2 to show that when $a \in \mathbb{Z}$ is such that $-a$ is not 2 or a perfect square, $f(x) = x^2 + a$ has the property that the splitting field of f^{n+1} is a degree-2^{2^n} extension of the splitting field of f^n for all sufficiently large n (Proposition 6.1). This has also been proved in [*W. Hindes*, Acta Arith. 159, No. 2, 149–167 (2013; Zbl 1296.14017)] by a different method, also assuming the *abc* conjecture.

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MSC:

37P05	Arithmetic and non-Archimedean dynamical systems involving polynomial and rational maps
11G50	Heights
14G25	Global ground fields in algebraic geometry

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