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Formality and hard Lefschetz property of aspherical manifolds. (English) Zbl 1283.53068

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A celebrated result of *P. Deligne*, *P. Griffiths*, *J. Morgan* and *D. Sullivan* asserts that compact Kähler manifolds are formal [Invent. Math. 29(3), 245–274 (1975; Zbl 0312.55011)]. Formality is a topological property which belongs to the realm of rational homotopy theory. It is well known that, on a compact Kähler manifold, the Lefschetz map is an isomorphism. Both formality and the hard Lefschetz property have been extensively used in order to produce examples of compact symplectic manifolds which carry no Kähler structure.

A solvmanifold is a compact quotient G/Γ of a simply connected solvable Lie group G by a lattice Γ. Any solvmanifold G/Γ is an Eilenberg-MacLane space $K(\Gamma, 1)$; in particular, a solvmanifold is not simply connected. Nilmanifolds, i.e. quotients of simply connected nilpotent Lie groups by a lattice, are a special case of solvmanifolds. *K. Hasegawa* [Proc. Am. Math. Soc. 106, No. 1, 65–71 (1989; Zbl 0691.53040)] showed that a nilmanifold is formal if and only if it is diffeomorphic to a torus. *C. Benson* and *C. S. Gordon* [Topology 27, No. 4, 513–518 (1988; Zbl 0672.53036)] proved that a symplectic nilmanifold satisfies the hard Lefschetz property if and only if it is diffeomorphic to a torus.

In this nice paper the author studies formality and the hard Lefschetz property in the context of solvmanifolds. He gives a sufficient condition for a solvmanifold G/Γ to be formal and to satisfy the hard Lefschetz property. Such condition is that the unipotent hull of Γ is abelian, and is shown to be equivalent to Γ being a finite extension of a lattice in a solvable group of the form $\mathbb{R}^n \ltimes_\phi \mathbb{R}^m$, where the $\mathbb{R}^n$-action on $\mathbb{R}^m$ is semisimple.

It was proved in [*O. Baues* and *V. Cortés*, Geom. Dedicata 122, 215–229 (2006; Zbl 1128.53043)] that a compact aspherical manifold with virtually polycyclic fundamental group Γ admits a Kähler structure if and only if Γ is virtually abelian. There is a close relationship between virtually polycyclic groups and lattices in solvable Lie groups: a lattice in a simply connected solvable Lie group is torsion-free and polycyclic, and every polycyclic group admits a finite index normal subgroup which is isomorphic to a lattice in a simply connected solvable Lie group. A simply connected solvable Lie group admits a virtually nilpotent lattice if and only if the eigenvalues of the adjoint representation Ad_g have absolute value 1.

By applying these facts to his results, the author provides examples of compact symplectic solvmanifolds which are formal and hard Lefschetz but carry no Kähler structure.

He also proves that the manifolds constructed in [*K. Oeljeklaus* and *M. Toma*, Ann. Inst. Fourier 55, No. 1, 161–171 (2005; Zbl 1071.32017)] are formal. Such manifolds are important because they disprove a conjecture of Vaisman on the topology of locally conformal Kähler manifolds.

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MSC:

53C55 Global differential geometry of Hermitian and Kählerian manifolds
20F16 Solvable groups, supersolvable groups
22E40 Discrete subgroups of Lie groups
55P62 Rational homotopy theory
32J27 Compact Kähler manifolds: generalizations, classification
53C30 Differential geometry of homogeneous manifolds

Keywords:

solvmanifold; symplectic non-Kähler manifold; formality; hard Lefschetz property

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