Schwede, Stefan

Algebraic versus topological triangulated categories. (English) Zbl 1221.18011

Holm, Thorsten (ed.) et al., Triangulated categories. Based on a workshop, Leeds, UK, August 2006. Cambridge: Cambridge University Press (ISBN 978-0-521-74431-7/pbk). London Mathematical Society Lecture Note Series 375, 389-407 (2010).

This expository paper draws some systematic differences between algebraic and topological triangulated categories. A triangulated category is algebraic in the sense of Keller if it is triangle equivalent to the stable category of a Frobenius category, i.e., an exact category in with enough projectives and in which injectives and projectives coincide. The most commonly known examples of algebraic triangulated categories include homotopy categories and derived categories of rings and the stable module categories of group algebras. A topological triangulated category is a triangulated category which is equivalent to the full triangulated subcategory of the homotopy category of a stable model category. Topological triangulated categories have their origin in stable homotopy theory and here the source categories are "non-additive" before passing to homotopy. The Spanier-Whitehead category is a prime example to keep in mind.

It turns out that there are properties defined entirely in terms of the triangulated category which hold in all algebraic examples but fails in some topological examples. For example, it is shown that if $\mathcal{T}$ is any algebraic triangulated category, and X is an object in $\mathcal{T}$, then $n \cdot X/n = 0$, where X/n is the cofibre of the n-fold multiple $n\ Id_X$ of the identity map Id_X on X. In contrast, it is shown that the same thing is not always true in a topological triangulated category. For instance, it is shown that $2 \cdot S/2 \neq 0$ in the Spanier-Whitehead category, where S is the sphere spectrum. On the other hand, for an odd prime p, it is shown that $p \cdot X/p = 0$ in any topological triangulated category. Thus the possibility of having $n \cdot X/n \neq 0$ in a topological triangulated category is really a 2-local phenomenon.

The paper has some interesting open problems which merit further study. For instance, the paper raises the following natural question. Does there exists a triangulated category $\mathcal{T}$ and an odd prime p such that $p \cdot X/p \neq 0$ for some object X in $\mathcal{T}$? The differences between algebraic and topological triangulated categories mentioned in this paper are all torsion phenomena, and when viewed rationally there is no difference between them.

For the entire collection see [Zbl 1195.18001].

Reviewer: Sunil K. Chebolu (Normal, IL)

MSC:

18E30 Derived categories, triangulated categories (MSC2010)
55P42 Stable homotopy theory, spectra

Full Text: arXiv