Maier, Robert S.

On rationally parametrized modular equations. (English) Zbl 1214.11049

J. Ramanujan Math. Soc. 24, No. 1, 1-73 (2009).

If the modular curve $X_0(M)$ associated with the modular group $\Gamma_0(M)$ is of genus 0, then its function field is generated by a function $t_M(\tau)$. This function t_M is called a Hauptmodul of $\Gamma_0(M)$. Further, for an integer N, if $X_0(MN)$ is of genus 0, then t_M and $t'_M = t_M(N\tau)$ are rationally expressed by a Hauptmodul t_{MN} of $\Gamma_0(MN)$. By this fact, the author obtains the level N modular equation between t_M and t'_M rationally parametrized by t_{MN}. Especially in the case $M = 1$ and $t_1 = j$, for every integer N such that $X_0(N)$ is of genus 0, a rational parameterization of the classical modular equation of level N is explicitly computed. For example, the classical modular equation of level 2 has a parameterization: $j = (t_2 + 16)^3/t_2, j' = j(2\tau) = (t_2 + 256)^3/t_2^2$. He defines functions $h_M(t_M)$ of t_M which are solutions of Picard-Fuchs equation, of hypergeometric, Heun or more general type, which are considered to be periods of elliptic curve parametrized by t_M. As a function of τ, $\mathfrak{h}(\tau) = h_M(t_M(\tau))$ is a modular form of weight one with respect to $\Gamma_0(M)$. From the parametrized equation of t_M and t'_M by t_{MN}, the author obtains various modular equations (algebraic transformations) of h_M by using pullbacks of Picard-Fuchs equations along the map: $X_0(MN) \to X_0(M)$ (viz. $t_{MN} \mapsto t_M$). The modular transformation of Ramanujan's elliptic integrals K_r of signatures $r = 2, 3, 4, 6$ are also obtained among these modular equations. The author points out that for $r = 2, 3, 4$, this gives a modern interpretation to Ramanujan's theories of integrals to alternative bases and that his theory of signature 6 turns out to fit into a general Gauss-Manin rather than a Picard-Fuchs framework. The functions t_M, h_M and $\mathfrak{h}_M(\tau)$ are given by eta products and the hypergeometric function ${}_2F_1$ and their explicit forms and modular equations are listed in 19 tables.

Reviewer: Noburo Ishii (Osaka)

MSC:

11F03 Modular and automorphic functions
11F20 Dedekind eta function, Dedekind sums
33C05 Classical hypergeometric functions, ${}_2F_1$

Keywords:
modular equation; Ramanujan; elliptic integral

Full Text: arXiv