Towers, John D.

Finite difference methods for approximating Heaviside functions. (English) Zbl 1171.65014

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The paper presents two algorithms to evaluate the integral $\mathcal{I} := \int_\Omega f(\vec{x})d\vec{x}$, where $\vec{x} \in \mathbb{R}^n$, $\Omega = \{\vec{x} : u(\vec{x}) > 0\}$ and $\partial\Omega = \{\vec{x} : u(\vec{x}) = 0\}$ is a compact manifold of codimension one. The method to be employed is motivated by the expression $\mathcal{I} = \int_{\mathbb{R}^n} H(u(\vec{x}))f(\vec{x})d\vec{x}$, where H is the Heaviside function. The approach consists of approximating H by finite differencing its first few primitives, a technique already used by the author to approximate delta functions.

A brief presentation of the algorithms is given below. Let

$$I(z) = \int_0^z H(\zeta)d\zeta \quad \text{and} \quad J(z) = \int_0^z I(\zeta)d\zeta.$$

The following relationships are derived:

$$I(u) = \langle \nabla J(u), \nabla u\rangle / |\nabla u|^2, \quad H(u) = \langle \nabla I(u), \nabla u\rangle / |\nabla u|^2,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product. By discretizing $H(u)$ the one-step algorithm $FDMH_1$ is obtained which converges at a rate of $\mathcal{O}(h^2)$ when u is smooth enough. By discretizing both relationships the two-step algorithm $FDMH_2$ is derived which can converge at a rate of $\mathcal{O}(h^3)$. These results are validated by means of some numerical examples.

Reviewer: Jesus Illán González (Vigo)

MSC:

65D32 Numerical quadrature and cubature formulas
41A55 Approximate quadratures
41A25 Rate of convergence, degree of approximation
41A63 Multidimensional problems (should also be assigned at least one other classification number from Section 41-XX)

Keywords:

Heaviside function; level set method; quadrature; irregular region; singular source term; finite difference; regular grid; convergence rate; one-step algorithm; two-step algorithm; numerical examples

Full Text: DOI

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