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**Affine Demazure modules and $T$-fixed point subschemes in the affine Grassmannian.** (English)

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Adv. Math. 221, No. 2, 570-600 (2009).

A geometrical proof of the Frenkel-Kac-Segal isomorphism is given. In the following, $G$ will be always a simple, connected algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Let $\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ be the associated untwisted affine Kac-Moody algebra. Given a positive integer $k$, let $\mathbb{V}(k\Lambda) = Ind^{\hat{\mathfrak{g}}}_{\mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}K}\mathbb{C}$ be the level $k$ module of $\hat{\mathfrak{g}}$ and let $\mathbb{L}(k\Lambda)$ be its unique irreducible quotient; $\mathbb{V}(k\Lambda)$ and $\mathbb{L}(k\Lambda)$ have a structure of vertex algebra. Let $\mathfrak{t}\subset\mathfrak{g}$ be a Cartan algebra and let $\hat{\mathfrak{t}}\subset\hat{\mathfrak{g}}$ be the associated Heisenberg Lie algebras. Let $V_{R_G}$ be $\bigoplus_{\lambda\in R_G}\pi_\lambda$ where $R_G$ is the coroot lattice of $\mathfrak{g}$ and $\pi_\lambda$ is the Fock module of $\mathfrak{t}$ with highest weight $\iota\lambda$. If $\mathfrak{g}$ is simply-laced, then $V_{R_G}$ have a structure of vertex algebras and the Frenkel-Kac-Segal states that, given a simple-laced simple algebra $\mathfrak{g}$, (1) $V_{R_G}\cong\mathbb{L}(\Lambda)$ as vertex algebras. In particular, they are isomorphic as $\hat{\mathfrak{t}}$-modules.

The author consider the following geometrical interpretation. Let $Gr_G = \mathcal{G}_\mathcal{K}/\mathcal{G}_\mathcal{O}$ be the affine Grassmannian of $G$, where $\mathcal{G}_\mathcal{K}$ is the group of maps from the punctured disc to $G$ and $\mathcal{G}_\mathcal{O}$ is the group of maps from the disc to $G$. If $G$ is simply-connected then the Picard group of $Gr_G$ is generated by an ample invertible sheaf $\mathcal{L}_G$. Then the Borel-Weyl theorem for affine Kac-Moody algebra identifies $\mathbb{L}(k\Lambda)$ (resp. $V_{R_G}$) with $H^0(Gr_G,\mathcal{L}_G^{\otimes k})^*$ (resp. $H^0(Gr_G,\mathcal{O}_{Gr_T}\otimes\mathcal{L}_G)^*$). Thus $\mathbb{L}(\Lambda)\cong V_{R_G}$ as $\hat{\mathfrak{t}}$-module if and only if the restriction morphism $\varphi:\mathcal{L}_G\to\mathcal{O}_{Gr_T}\otimes\mathcal{L}_G$ induces an isomorphism between the spaces of global sections. If $G$ is not simple connected, then $Gr_G$ is not connected and the author define $\mathcal{L}_G$ as the line bundle whose restriction to each connected component is the ample generator of its Picard group. The Borel-Weyl theorem hold again (see Proposition 1.4.4). Recall that $Gr_G$ is stratified by $G_\mathcal{O}$-orbits, $\{Gr_G^\lambda\}$, indexed by the dominant coweights $\{\lambda\}$. Moreover, $Gr_G = \varinjlim \overline{Gr}_G^\lambda$, where the Schubert varieties $\overline{Gr}_G^\lambda$ are defined as the closures of the $Gr_G^\lambda$.

The maximal torus $T$ of $G$ acts on the Schubert varieties and the natural embedding $Gr_T\subset Gr_G$ identifies $Gr_T$ with the $T$-fixed point scheme of $Gr_G$ (see §1.3). Moreover $Gr_T\times_{Gr_G}\overline{Gr}_G^\lambda$ is the $T$-fixed point scheme of $\overline{Gr}_G^\lambda$. The main theorem of this article states that the restriction of $\varphi$ to $\overline{Gr}_G^\lambda$ induces an isomorphism on the global sections, (2) $H^0(\overline{Gr}_G^\lambda,\mathcal{L}_G)\to H^0(\overline{Gr}_G^\lambda,\mathcal{O}_{(\overline{Gr}_G^\lambda)^T}\otimes\mathcal{L}_G)$, if $G$ has type A or D. Furthermore, the same fact holds for many coweights if $G$ has type E. In many parts of the proof it is used that $(\overline{Gr}_G^\lambda)^T$ is a finite scheme.

The difficult part of the proof is showing the injectivity of (2). Indeed it is proved that the restriction of $\mathcal{L}_G^k$ from $\overline{Gr}_G^\lambda$ to $(\overline{Gr}_G^\lambda)^T$ induces a surjective morphism on the global sections for any simple algebraic group $G$.

The first step to prove the injectivity is a reduction to the case of fundamental dominant coweights. Given two dominant coweights $\lambda$ and $\mu$, the author constructs a family of varieties with generic fibre is $\overline{Gr}_G^\lambda\times\overline{Gr}_G^\mu$ while the special fibre is $\overline{Gr}_G^{\lambda+\mu}$, using the following result on the Demazure affine modules (here $\widetilde{G}$ is the simply connected cover of $G$): $H^0(\overline{Gr}_G^{\lambda+\mu},\mathcal{L}_G^k)\cong H^0(\overline{Gr}_G^\lambda,\mathcal{L}_G^k)\otimes H^0(\overline{Gr}_G^\mu,\mathcal{L}_G^k)$ as $\widetilde{G}$-modules. The author includes a proof of this isomorphism which is more geometrical than the original one. To prove the reduction step, the author uses the this family together with the interpretation of the affine Grassmannian as a module space over a smooth curve.

Next, he prove that the Schubert variety $\overline{Gr}_G^\lambda$ contains many rational curve with known degree. If $\lambda$ is minuscule (and $G$ is simply laced) then these curves have degree one. The author proves the injectivity by showing that an arbitrarily fixed section of $H^0(\overline{Gr}_G^\lambda,\mathcal{I}^\lambda(1))$ is zero over certain $T$-invariant subvarieties $Z$ (by induction on the dimension of $Z$). Here $\mathcal{I}^\lambda\subset\mathcal{O}_{\overline{Gr}_G^\lambda}$ is the ideal sheaf defining $(\overline{Gr}_G^\lambda)^T$. Moreover, the previous subvarieties includes all the $T$-invariant curves and the whole Schubert variety. Therefore, the case of type A is proved.


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Next, he prove the injectivity when $\lambda$ is a longest root. The proof is similar, but he need to consider also curve of degree 2. Moreover he uses the result for $G = SL_2$. This fact proves the case of $D_4$.

The case of $D_n$ is proved by induction on $n$ and by induction on $i$, where $\lambda$ is $i$-th fundamental coweight and the weight are indexed according to [*N. Bourbaki*, Groupes et algébres de Lie. Chapitres 4, 5 et 6. Elements de Mathematique. (Paris) etc.: Masson. (1981; Zbl 0483.22001)]. He need also the isomorphism (2) for the case A. Finally, the author can prove some other cases when the type of $G$ is $E$ and he conjectures that the map is always injective for the type $E$.

It is necessary to note that the author can reprove the FKS-isomorphism for all the simply laced group. The isomorphism (1) as $\hat{\mathfrak{t}}$-modules clearly follows from (2) for the type A and D. To prove the case $E$ the author show that also in this case any connected components of $Gr_G$ is the direct limits of Schubert varieties associated to weight for which (2) is an isomorphism.

To prove that (1) is an isomorphism of vertex algebras he uses the languages of Kac-moody factorization algebras. Finally, he prove an identification of the modules over $V_{R_G}$ with the modules over $\mathbb{L}(\Lambda)$.

Reviewer: Alessandro Ruzzi (Roma)

**MSC:**

14M15 Grassmannians, Schubert varieties, flag manifolds
17B67 Kac-Moody (super)algebras; extended affine Lie algebras; toroidal Lie algebras
17B69 Vertex operators; vertex operator algebras and related structures

Cited in **9** Documents

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basic representation; Frenkel-Kac-Segal isomorphism; affine Grassmannian

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**References:**

[1] Beauville, A.; Laszlo, Y., Conformal blocks and generalized theta functions, Comm. math. phys., 164, 385-419, (1994) · Zbl 0815.14015

[2] Beauville, A.; Laszlo, Y., Un lemme de descente, C. R. acad. sci. Paris Sér. I math., 320, 335-340, (1995) · Zbl 0852.13005

[3] Beauville, A.; Laszlo, Y.; Sorger, C., The Picard group of the moduli of \textit{G}-bundles on a curve, Compos. math., 112, 183-216, (1998) · Zbl 0976.14024

[4] Beilinson, A., Langlands parameters for Heisenberg modules, (), 51-60 · Zbl 1195.17020

[5] Beilinson, A.; Drinfeld, V., Chiral algebras, Amer. math. soc. colloq. publ., vol. 51, (2004), Amer. Math. Soc. Providence, RI · Zbl 1138.17300

[6] Beilinson, A.; Drinfeld, V., Quantization of Hitchin's integrable system and Hecke eigensheaves, preprint, available at · Zbl 0864.14007

[7] Bourbaki, N., Groupes et algèbres de Lie, chapitres IV-VI, (1968), Hermann Paris · Zbl 0186.33001

[8] Demazure, M.; Gabriel, P., Groupes algébriques, (1970), North-Holland Amsterdam

[9] Dong, C., Vertex algebras associated with even lattices, J. algebra, 161, 245-265, (1993) · Zbl 0807.17022

[10] Evens, S.; Mirković, I., Characteristic cycles for the loop Grassmannian and nilpotent orbits, Duke math. J., 97, 1, 109-126, (1999) · Zbl 1160.22306

[11] Faltings, G., Algebraic loop groups and moduli spaces of bundles, J. eur. math. soc., 5, 41-68, (2003) · Zbl 1020.14002

[12] Fourier, G.; Littelmann, P., Tensor product structure of affine Demazure modules and limit constructions, Nagoya math. J., 182, 171-198, (2006) · Zbl 1143.22010

[13] Frenkel, E.; Ben-Zvi, D., Vertex algebras and algebraic curves, Math. surveys monogr., vol. 88, (2004), Amer. Math. Soc. Providence, RI · Zbl 1106.17035

[14] Frenkel, I.; Kac, V., Basic representations of affine Lie algebras and dual resonance models, Invent. math., 62, 23-66, (1980) · Zbl 0493.17010

[15] Gaitsgory, D., Notes on 2D conformal field theory and string theory, (), 1017-1089 · Zbl 1170.81429

[16] Goresky, M.; Kottwitz, R.; Macphserson, R., Homology of affine Springer fibers in the unramified case, Duke math. J., 121, 3, 509-561, (2004) · Zbl 1162.14311

[17] Grothendieck, A.; Dieudonné, J., Eléments de géométrie algébrique III, (), 5-91

[18] Hartshorne, R., Algebraic geometry, (1977), Springer-Verlag Berlin · Zbl 0367.14001

[19] Kac, V., Infinite-dimensional algebras, Dedekind's \textit{$\eta$}-function, classical Möbius function and the very strange formula,

Adv. math. (30), 85-136, (1978) · Zbl 0391.17010

[20] Kac, V.; Kazhdan, D.; Lepowsky, J.; Wilson, R., Realization of the basic representations of the Euclidean Lie algebras, Adv. math., 42, 1, 83-112, (1981) · Zbl 0476.17003

[21] Kac, V.; Peterson, D., 112 constructions of the basic representation of the loop group of $E_8$, (), 276-298

[22] Kapranov, M.; Vasserot, E., Formal loops II: A local Riemann-Roch theorem for determinantal gerbes, Ann. sci. école norm. sup. (4), 40, 1, 113-133, (2007) · Zbl 1129.14022

[23] Kazhdan, D.; Lusztig, G., Fixed point varieties on affine flag manifolds, Israel J. math., 62, 2, 129-168, (1988) · Zbl 0658.22005

[24] Kumar, S., Demazure character formula in arbitrary Kac-Moody setting, Invent. math., 89, 395-423, (1987) · Zbl 0635.14023

[25] Laszlo, Y.; Sorger, C., The line bundles on the moduli of parabolic \textit{G}-bundles over curves and their sections, Ann. sci. école norm. sup. (4), 30, 4, 499-525, (1997) · Zbl 0918.14004

[26] Malkin, A.; Ostrik, V.; Vybornov, M., The minimal degeneration singularities in the affine Grassmannians, Duke math. J., 126, 2, 233-249, (2005) · Zbl 1078.14016

[27] Mathieu, O., Formules de caractères pour LES algèbres de Kac-Moody générales, Astérisque, 159-160, (1988)

[28] Mehta, V.B.; Ramanathan, A., Frobenius splitting and cohomology vanishing for Schubert varieties, Ann. of math. (2), 122, 1, 27-40, (1985) · Zbl 0601.14043

[29] Mirković, I.; Vilonen, K., Geometric Langlands duality and representations of algebraic groups over commutative rings, Ann. of math. (2), 166, 1, 95-143, (2007) · Zbl 1138.22013

[30] Segal, G., Unitary representations of some infinite-dimensional groups, Comm. math. phys., 80, 3, 301-342, (1981) · Zbl 0495.22017

[31] X. Zhu, Basic representations via affine Springer fiber, in preparation

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