Guo, Ben-Yu; **Shen, Jie**; **Wang, Li-Lian**

Generalized Jacobi polynomials/functions and their applications. (English) Zbl 1171.33006
Appl. Numer. Math. 59, No. 5, 1011-1028 (2009).

For arbitrary real numbers α, β the authors define the generalized Jacobi polynomials/functions $j_n^{\alpha,\beta}$ by

$$j_n^{\alpha,\beta}(x) = \omega^{\hat{\alpha},\hat{\beta}}(x)\, J_{n_1}^{\tilde{\alpha},\tilde{\beta}}(x), \quad n \geq n_0^{\alpha,\beta},\ -1 < x < 1,$$

where

$$\hat{\alpha} = \begin{cases} -\alpha, & \alpha \leq -1 \\ 0, & \alpha > -1 \end{cases}; \quad \tilde{\alpha} = \begin{cases} -\alpha, & \alpha \leq -1 \\ \alpha & \alpha > -1 \end{cases}, \quad \omega^{a,b}(x) = (1-x)^a(1+x)^b$$

and

$$n_0 = n_0^{\alpha,\beta} := [\hat{\alpha}] + [\hat{\beta}],\ n_1 = n_1^{\alpha,\beta} := n - n_0^{\alpha,\beta}.$$

([·] the greatest integer function)

For α, $\beta > -1$ and for α, β negative integers they are polynomials of degree n, coinciding—up to a multiplicative constant—with the definition in *G. Szegő* [Orthogonal Polynomials, AMS (1975; Zbl 0305.42011)].

Other values of the parameters lead to

$$j_n^{\alpha,\beta}(x) = \begin{cases} (1-x)^{-\alpha}(1+x)^{-\beta} J_{n_1}^{-\alpha,-\beta}(x), & \alpha,\beta \leq -1;\ n_1 = n - [-\alpha] - [-\beta] \\ (1-x)^{-\alpha} J_{n_1}^{-\alpha,\beta}(x), & \alpha \leq -1, \beta > -1;\ n_1 = n - [-\alpha] \\ (1+x)^{-\beta} J_{n_1}^{\alpha,-\beta}(x), & \alpha > -1, \beta \leq -1;\ n_1 = n - [-\beta]. \end{cases}$$

The GJP/Fs satisfy orthogonality properties, a Sturm-Liouville equation, derivative recurrence relations, approximation properties on the underlying Sobolev space, etc.

As an application spectral Galerkin methods for higher order differential equations are studied, including error estimates and, furthermore, some numerical results are given for the equation

$$u^{(6)}(x) - u(x) = f(x)$$

on $(-1, 1)$ with boundary conditions $u(\pm 1), u'(\pm 1), u''(\pm 1)$ and driving force $f(x)$ such that the exact solution is

$$u(x) = (1-x)e^x$$

and

$$u(x) = (1+x)^p e^x.$$

Reviewer: Marcel G. de Bruin (Haarlem)

MSC:

33C45 Orthogonal polynomials and functions of hypergeometric type (Jacobi, Laguerre, Hermite, Askey scheme, etc.)
35J05 Laplace operator, Helmholtz equation (reduced wave equation), Poisson equation
42C05 Orthogonal functions and polynomials, general theory of nontrigonometric harmonic analysis
65N22 Numerical solution of discretized equations for boundary value problems involving PDEs

Keywords:

Jacobi polynomials; spectral approximation; error estimate; high-order differential equations

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