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A spectral Galerkin method for the coupled Orr-Sommerfeld and induction equations for free-surface MHD. (English) [Zbl 1330.76055](#)

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Summary: We develop and test spectral Galerkin schemes to solve the coupled Orr-Sommerfeld and induction equations for parallel, incompressible MHD in free-surface and fixed-boundary geometries. The schemes' discrete bases consist of Legendre internal shape functions, supplemented with nodal shape functions for the weak imposition of the stress and insulating boundary conditions. The orthogonality properties of the basis polynomials solve the matrix-coefficient growth problem, and eigenvalue-eigenfunction pairs can be computed stably at spectral orders at least as large as $p = 3000$ with p -independent roundoff error. Accuracy is limited instead by roundoff sensitivity due to non-normality of the stability operators at large hydrodynamic and/or magnetic Reynolds numbers ($Re, Rm \gtrsim 4 \times 10^4$). In problems with Hartmann velocity and magnetic-field profiles we employ suitable Gauss quadrature rules to evaluate the associated exponentially weighted sesquilinear forms without error. An alternative approach, which involves approximating the forms by means of Legendre-Gauss-Lobatto quadrature at the $2p - 1$ precision level, is found to yield equal eigenvalues within roundoff error. As a consistency check, we compare modal growth rates to energy growth rates in nonlinear simulations and record relative discrepancy smaller than 10^{-5} for the least stable mode in free-surface flow at $Re = 3 \times 10^4$. Moreover, we confirm that the computed normal modes satisfy an energy conservation law for free-surface MHD with error smaller than 10^{-6} . The critical Reynolds number in free-surface MHD is found to be sensitive to the magnetic Prandtl number Pm , even at the $Pm = O(10^{-5})$ regime of liquid metals.

MSC:

- 76E25 Stability and instability of magnetohydrodynamic and electrohydrodynamic flows
- 76M10 Finite element methods applied to problems in fluid mechanics
- 76W05 Magnetohydrodynamics and electrohydrodynamics
- 65L60 Finite element, Rayleigh-Ritz, Galerkin and collocation methods for ordinary differential equations
- 76M25 Other numerical methods (fluid mechanics) (MSC2010)

Cited in **8** Documents

Keywords:

eigenvalue problems; spectral Galerkin method; hydrodynamic stability; Orr-Sommerfeld equations; free-surface MHD

Software:

ARPACK; Eigtool; LAPACK

Full Text: [DOI](#)

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