Jones, Rafe

The density of prime divisors in the arithmetic dynamics of quadratic polynomials. (English)

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Let $a_1, a_2, \dots$ be an integer sequence. We say that a prime p divides a sequence S if it divides at least one non-zero element of the sequence. The set of such primes we denote by $P(S)$. By $D(S)$ we denote the natural density of primes p dividing the sequence if it exists. The author is interested in $D(S)$ for sequences S that are defined recursively by $a_n = f(a_{n-1}, \dots, a_{n-k})$, which is said to be of order k. Examples are Fibonacci numbers, Lucas numbers and Mersenne numbers. In case the recurrence is linear there is a considerable body of literature devoted to determining $D(S)$. For example, if $a_n = 2^n + 1$, then Hasse showed in 1966 that $D(S) = 17/24$. In contrast the case where f is not a linear function is little studied, even in the case $k = 1$, which we assume from now on. The main result is that of *R. W. K. Odoni* [Proc. Lond. Math. Soc. (3) 51, 385–414 (1985; Zbl 0622.12011)] who showed that 'most' polynomials f of degree d have the property that all of their integer orbits contain a 'very small' proportion of the primes. Since $k = 1$, we may write $P(f, a_0)$ instead of $P(S)$. Odoni's result does not determine $D(P(f, a_0))$.

The main concern of the author is to determine $D(P(f, a_0))$ for various f and a_0. His main result is that $D(P(f, a_0)) = 0$ for the following infinite families: $x^2 - kx + k$, $x^2 + kx - 1$, $x^2 + k$ and $x^2 - 2kx + k$, where for the latter three families one has to exclude the following k-sets: $\{0, 2\}$, $\{-1\}$, respectively $\{\pm 1\}$. In case $k = 1$ for the first family, this generalizes earlier work by *R. W. K. Odoni* [J. Lond. Math. Soc. (2) 32, 1–11 (1985; Zbl 0574.10020)]. A more general result is proved for sequences of the form $a_n = g \circ f^n(a_0)$, where f and g are such that $g \circ f^n(x)$ is irreducible for all n, and satisfy certain hypotheses involving either finiteness of the set $\{f^n(0) : n = 1, 2, \dots\}$ or strong divisibility properties of the forward orbit of the critical point of f. The assumption that $g \circ f^n(x)$ is irreducible for all n is not a major hypothesis, and the author devotes one section of the paper to giving new sufficient conditions for this to hold. The arithmetic nature of the forward orbit of the critical point plays an essential role in allowing for a proof of these results, mirroring its importance in the complex and real dynamics of quadratic maps.

The method of proof revolves around a study of the Galois towers generated by the polynomials $g(f^n))$ for $n = 1, 2, \dots$. To accomplish this, the author amplifies techniques from the theory of stochastic processes used by him in an earlier paper [Compos. Math. 143, No. 5, 1108–1126 (2007; Zbl 1166.11040)]. Indeed, the author uses a spectrum of methods much wider than usually employed in this area, making this an impressive and important paper in the subject of density of prime divisors of sequences.

Reviewer: Pieter Moree (Bonn)

MSC:

37P05	Arithmetic and non-Archimedean dynamical systems involving polynomial and rational maps
11B37	Recurrences
37E99	Low-dimensional dynamical systems
60G42	Martingales with discrete parameter

Keywords:
arithmetic dynamics; critical point; density of prime divisors; recurrence; martingales; Galois towers

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