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Sharp embeddings of Besov spaces involving only logarithmic smoothness. (English)

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The paper contains sharp embedding assertions for spaces $B^{0,\beta}_{p,r}(\mathbb{R}^n)$ of Besov type into spaces $L^{\text{loc}}_{p,q;\gamma}(\mathbb{R}^n)$ of Lorentz-Zygmund type. The Besov spaces are defined by means of the modulus of continuity $\omega_1(f,t)_p$,

$$\|f|B^{0,\beta}_{p,r}(\mathbb{R}^n)\| = \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^1 (1+|\ln t|)^{\beta r}\omega_1^r(f,t)_p \frac{dt}{t}\right)^{1/r},$$

whereas the Lorentz-Zygmund spaces $L^{\text{loc}}_{p,q;\gamma}(\mathbb{R}^n)$ collect all f with finite quasi-norm

$$\left(\int_0^1 t^{q/p}(1+|\ln t|)^{\gamma q} f^*(t)^q \frac{dt}{t}\right)^{1/q},$$

where $1\le p<\infty$, $1\le r,q\le\infty$, $\beta,\gamma\in\mathbb{R}$ with $\beta+1/r>0$, and f^* denotes the non-increasing rearrangement, as usual. The essential point here is that the Besov spaces have zero classical smoothness and differ from their counterparts defined by Fourier analytical methods, say. Thus many of the standard approaches to prove optimality or sharpness of embeddings do not work and the essential benefit of this paper is to close this gap and, moreover, find new phenomena in connection with growth envelopes and limiting embeddings.

The main results, listed in Section 3 of this interesting paper, contain criteria for the above embedding of the form that $B^{0,\beta}_{p,r}(\mathbb{R}^n)$ is embedded into $L^{\text{loc}}_{p,q;\gamma}(\mathbb{R}^n)$ with $\gamma=\beta+\frac{1}{r}+\frac{1}{\max(p,q)}-\frac{1}{q}$ if, and only if, $q\ge r$ (Theorem 3.1). Furthermore, the growth envelope of the space $B^{0,\beta}_{p,r}(\mathbb{R}^n)$ is determined by the pair $(t^{-\frac{1}{p}}(1+|\ln t|)^{-\beta-\frac{1}{r}},\max(p,r))$.

The proofs, presented in Section 5-9, rely on an inequality by *V. I. Kolyada* [Math. USSR, Sb. 64, No. 1, 1–21 (1989); translation from Mat. Sb., Nov. Ser. 136(178), No. 1(5), 3–23 (1988; Zbl 0693.46030)] and some inverse of it, stated as Proposition 3.5 and proved in this paper in Section 4. It connects the modulus of continuity and the non-increasing rearrangement in a tricky way and is certainly of separate interest.

Reviewer: Dorothee Haroske (Jena)

MSC:

46E35 Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems
46E30 Spaces of measurable functions (L^p-spaces, Orlicz spaces, Köthe function spaces, Lorentz spaces, rearrangement invariant spaces, ideal spaces, etc.)
26D10 Inequalities involving derivatives and differential and integral operators

Keywords:

Besov spaces with generalized smoothness; Lorentz-Zygmund spaces; sharp embeddings; growth envelopes

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References:

[1] Caetano, A.; Leopold, H.-G., Local growth envelopes of Triebel-Lizorkin spaces of generalized smoothness, J. Fourier anal. appl., 12, 427-445, (2006) · Zbl 1122.46017

[2] Caetano, A.M.; Farkas, W., Local growth envelopes of Besov spaces of generalized smoothness, Z. anal. anwendungen, 25, 265-298, (2006) · Zbl 1119.46032

[3] Caetano, A.M.; Moura, S.D., Local growth envelopes of spaces of generalized smoothness: the critical case, Math. inequality appl., 7, 573-606, (2004) · Zbl 1076.46025

[4] Caetano, A.M.; Moura, S.D., Local growth envelopes of spaces of generalized smoothness: the sub-critical case, Math. nachr., 273, 1, 43-57, (2004) · Zbl 1070.46020

[5] Gogatishvili, A.; Pick, L., Discretization and anti-discretization of rearrangement-invariant norms, Publ. mat., 47, 2, 311-358, (2003) · Zbl 1066.46023

[6] Gogatishvili, A.; Pick, L., Embeddings and duality theorems for weak classical Lorentz spaces, Canad. math. bull., 49, 1, 82-95, (2006) · Zbl 1106.26018

[7] Goldman, M., Embeddings of nikol'skij – besov spaces into weighted lorents spaces, Trudy mat. inst. steklova, 180, 93-95, (1987), (in Russian)

[8] Gol'dman, M.L.; Kerman, R., On optimal embedding of Calderón spaces and generalized Besov spaces, Trudy mat. inst. steklova, 243, 161-193, (2003), (in Russian) (English translation: Proc. Steklov Inst. Math. 243 (2003) 154-184) · Zbl 1090.46022

[9] Gurka, P.; Opic, B., Sharp embeddings of Besov spaces with logarithmic smoothness, Rev. mat. complutense, 18, 81-110, (2005) · Zbl 1083.46018

[10] Gurka, P.; Opic, B., Sharp embeddings of Besov-type spaces, J. comput. appl. math., 208, 235-269, (2007) · Zbl 1132.46022

[11] G. Hardy, J.E. Littlewood, G. Pólya, Inequalities, reprinted second ed., Cambridge University Press, Cambridge, 1994.

[12] D.D. Haroske, Envelopes in function spaces—a first approach, preprint Math/Inf/16/01, University of Jena, Germany, 2001 (cf. also D.D. Haroske, Envelopes and sharp embeddings of function spaces, Chapman \& Hall, CRC, Boca Raton, 2007).

[13] Kolyada, V., Estimates of rearrangements and embedding theorems, Mat. sb., 136, 3-23, (1988), (English transl. in Math. USSR-Sb. 64(1) (1989) 1-21)

[14] Netrusov, Yu., Theorems for the embedding of Besov spaces into ideal spaces, Zap. nauchn. sem. leningrad. otdel. mat. inst. Steklov. (LOMI), 159, 69-82, (1987), (Russian) (English translation: J. Soviet. Math. 47 (1989) 2871-2881) · Zbl 0636.46036

[15] Opic, B.; Kufner, A., Hardy-type inequalities, Pitman research notes in mathematics, vol. 219, (1990), Longman Science \& Technology Harlow

[16] Triebel, H., The structure of functions, (2001), Birkhäuser Basel