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On short exponential sums involving Fourier coefficients of holomorphic cusp forms. (English)

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From the text: We improve known estimates for the linear exponential sums containing Fourier coefficients of holomorphic cusp forms and show that in some cases, our bound is actually sharp. We also briefly visit nonlinear exponential sums, and prove some results concerning the density of rational numbers satisfying certain conditions.

Holomorphic cusp forms can be represented as Fourier series $F(z) = \sum_{n=1}^{\infty} a(n)n^{(\kappa-1)/2}e(nz)$, where the numbers $a(n)$ are called normalized Fourier coefficients and κ is the weight of the form. It is of interest to consider exponential sums of the normalized Fourier coefficients:

$$A(M,\Delta,\alpha) = \sum_{M\leq n\leq M+\Delta} a(n)e(n\alpha)$$

with $0 < \Delta \leq M$. *J. R. Wilton*'s estimate [Math. Proc. Camb. Philos. Soc. 25, 121–129 (1929; JFM 55.0709.02)]

$$\sum_{n\leq M} a(n)e(n\alpha) \ll M^{1/2}\log M$$

from the year 1929 is a classical result. It is nearly sharp, only the logarithm can be removed and that was done by *M. Jutila* [Proc. Indian Acad. Sci., Math. Sci. 97, No. 1–3, 157–166 (1987; Zbl 0658.10043)] in 1987. Before that *P. Deligne* [Publ. Math., Inst. Hautes Étud. Sci. 43, 273–307 (1974; Zbl 0287.14001)] proved the extremely useful bound $|a(n)| \leq d(n) \ll n^{\varepsilon}$. When the parameter α is rational the estimates are much better as Jutila showed ["Lectures on a method in the theory of exponential sums. " Bombay: Tata Institute of Fundamental Research (1987; Zbl 0671.10031)].

The situation changes drastically when one considers short sums. Jutila [Zbl 0658.10043] proved the estimate

$$\sum_{M\leq n\leq M+\Delta} a(n)e(n\alpha) \ll M^{1/2-a_1}(1+\Delta M^{-1/2})$$

with a_1 positive, and he also proved that

$$\sum_{M\leq n\leq M+\Delta} a(n)e(n\alpha) \ll M^{1/2}. \tag{1}$$

These results are a natural starting point for this article. We improve the results by showing that for, $1 \leq \Delta \ll M^{3/4}$,

$$A(M,\Delta,\alpha) \ll M^{1/2-f(\log_M \Delta)},$$

where f is positive when $\Delta \ll M^{3/4-\varepsilon}$ for any fixed $\varepsilon > 0$. As usual, Voronoi type sum formulas are very helpful in the proofs. Results for shorter sums lean heavily on good estimates for nonlinear sums. For longer sums an approximate functional equation for exponential sums and partial integration are crucial. We also give an explicit formula for f. In the final chapter we prove that

$$\left|A\left(M, M^{3/4}, \frac{1}{\sqrt{M}}\right)\right| \asymp M^{1/2}$$

and hence conclude Jutila's estimate (1) to be best possible when $M^{3/4} \ll \Delta$. To be precise, we derive this as a special case while proving that our estimate is best possible for $M^{3/4-1/32+\varepsilon} \ll \Delta$.

MSC:

11L07 Estimates on exponential sums
11F30 Fourier coefficients of automorphic forms

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