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**Generalizations of Frobenius' theorem on manifolds and subcartesian spaces.** (English)

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A Hausdorff differential space $S$ in the sense of Sikorski is called a subcartesian space if it is covered by open sets which are diffeomorphic to subsets of $\mathbb{R}^n$. A generalized distribution on a manifold $M$ is a subset $D$ of $TM$ such that $D_p = D \cap T_pM$ is a subspace of $T_pM$ for each $p \in M$. The dimension of $D_p$ is called the rank of $D$ at $p$. A generalized distribution is smooth if it is locally spanned by smooth vector fields. An integral manifold of a distribution $D$ on $M$ is an immersed submanifold $N$ of $M$ such that $T\iota_{NM}(T_qN) = D_{\iota(q)}$ for every $q \in N$, where $\iota_{NM} : N \hookrightarrow M$ is the inclusion map. A distribution $D$ on $M$ is said to be integrable if there exists an integral manifold of $D$ containing $p$ for every $p \in M$. If $D$ is integrable, then every integral manifold of $D$ can be extended to a maximal integral manifold of $D$. Let $\mathcal{D}$ be the family of all vector fields on $M$ with values in $D$. The distribution $D$ is said to be involutive if the family $\mathcal{D}$ is closed under the Lie bracket of vector fields. An integrable distribution is involutive.

The Theorem of Frobenius states that a constant rank distribution $D$ on a manifold $M$ is integrable if it is involutive, while the Theorem of Sussmann states that a distribution $D$ on a manifold $M$ is integrable if and only if it is preserved by local one-parameter groups of local diffeomorphisms of $M$ generated by vector fields on $M$ with values in $D$. I. Kolář, P. W. Michor, and J. Slovák proved that a distribution $D$ on a manifold $M$ is integrable if it is involutive and its rank is constant on integral curves of vector fields on $M$ with values in $D$. Frobenius' and Sussmann's theorems are special cases of this result. A distribution $D$ spanned by a family $\mathcal{F}$ of vector fields on $M$ is said to be involutive on an orbit $O$ of $\mathcal{F}$ if the Lie bracket $[X, Y](p) \in D_p$ for every $X, Y \in \mathcal{F}$ and every $p \in O$.

In this paper, the author shows that if $\mathcal{F}$ is a family of vector fields on a manifold or a subcartesian space $M$ spanning a distribution $D$, then an orbit $O$ of $\mathcal{F}$ is an integral manifold of $D$ if $D$ is involutive on $O$ and it has constant rank on $O$. This result implies the Theorem of Kolář, Michor, and Slovák, and hence it implies Frobenius' and Sussmann's Theorems.

Reviewer: Andrew Bucki (Edmond)

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58A40 Differential spaces

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