Silverman, Joseph H.
The arithmetic of dynamical systems. (English) Zbl 1130.37001
Graduate Texts in Mathematics 241. New York, NY: Springer (ISBN 978-0-387-69903-5/hbk). ix, 511 p. (2007).

The connections between dynamical systems and number theory arise in many different ways. One subject may provide a new insight or tool for the other; one may prove to be a source of significant examples for the other; analogies may motivate new developments; on occasion, the two subjects find they are looking at the same object from different perspectives. This remarkable book unifies and clarifies one of these connections, in the setting of what might be called 'Arithmetic Dynamical Systems'.

Viewed from dynamics, it is concerned with fundamental questions about the behaviour of points under iteration of a map for a special class of maps. Viewed from number theory, most of the content and many of the questions will touch on familiar themes from a novel viewpoint. From a broader perspective, this book gives a careful, detailed, coherent, well-motivated path through a beautiful part of mathematics, and opens the door to a field of active research. There are a huge number of exercises, ranging from routine arguments to practice using definitions to genuinely difficult problems. The formal background required is modest — algebraic number theory, valuations, and elementary complex analysis — making it suitable for many graduate students. The more advanced material is daunting without some background in algebraic geometry, but no prior knowledge from dynamical systems or advanced number theory is required.

The book is concerned with rational maps $\phi : \mathbb{P}^d(\mathbb{K}) \to \mathbb{P}^d(\mathbb{K})$ on d-dimensional projective space over a field $\mathbb{K}$, and sets itself the (seemingly) modest goal of classifying the elements of $\mathbb{P}^d(\mathbb{K})$ according to their behaviour under iteration of ϕ.

The classical case of 'complex dynamics' ($d = 1$, $\mathbb{K} = \mathbb{C}$) in Chapter 1 is used to introduce some of the major themes: equivalence of dynamical systems, the role of critical points and their relation to the Riemann–Hurwitz formula, periodic points and their multipliers, Julia and Fatou sets, and the important special case of rational maps associated to a group (Chebyshev polynomials, rational maps arising from elliptic curves, and the Lattés examples). The remaining chapters develop analogues of all these phenomena in various settings.

In Chapter 2, the case $d = 1$ and $\mathbb{K}$ a local nonarchimedean field is studied, and the key notion of good reduction is introduced (throughout the main emphasis is on characteristic zero, but sufficiently well-behaved fields of positive characteristic are included).

Chapter 3 develops the case $d = 1$ and $\mathbb{K}$ a global field, and another key notion is introduced: the use of heights (naive, canonical, local, morphic) to study dynamics. In this chapter some of the connections with Diophantine analysis are described, and one of the major framing conjectures in the subject (the Uniform Boundedness Conjecture of Morton and the author, which asks if the number of pre-periodic points of a morphism of $\mathbb{P}^d(\mathbb{K})$ is bounded uniformly by a constant depending only on the degree of the morphism, d, and the degree of the number field $\mathbb{K}$) is described.

Chapter 4 introduces families of dynamical systems and exposes deep analogies between moduli spaces of rational maps with marked periodic points and moduli spaces of abelian varieties.

Chapter 5 returns to the case $d = 1$ and $\mathbb{K}$ a local field, but in the more difficult setting of bad reduction. The chapter ends with an introduction to Berkovich space as a setting for studying the dynamics of rational maps over $\mathbb{C}_p$.

Chapter 6 picks up the special examples of rational maps associated to an underlying algebraic group and studies them more systematically. In order to make the book fairly self-contained, a brief introduction to elliptic curves is provided.

The final chapter introduces rational maps with $d > 1$, and includes an introduction to the algebraic geometry needed. By this stage the big picture analogy between rational (or integral) points on varieties and rational (or integral) points in the orbit of a point, between torsion on a variety and (pre-)periodic points, has been explored in great depth.

Of necessity this book has to refer to the literature for some proofs (this in part reflects the diverse nature of the subject: the quoted theorems range across dynamics both complex and p-adic, elliptic curves, Diophantine analysis, and so on). It also pursues a specifically arithmetic path (for example, 'entropy' has two entries in the index, one to a notion of algebraic entropy, 'mixing' does not appear – while various kinds of 'height' have several hundred entries). Thus there is negligible overlap with the existing literature on (algebraic) dynamical systems, ergodic theory of numbers and so on.

This book should be of great interest to anyone interested in dynamics or number theory, and will attract them into this fascinating field. Not for the first time, the mathematical community owes the author thanks for a wonderful book called 'The Arithmetic of...'.

Reviewer: Thomas Ward (Norwich)

MSC:

37-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to dynamical systems and ergodic theory
11-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to number theory
37F10 Dynamics of complex polynomials, rational maps, entire and meromorphic functions; Fatou and Julia sets
37C25 Fixed points and periodic points of dynamical systems; fixed-point index theory, local dynamics
11G50 Heights
11G05 Elliptic curves over global fields
11G07 Elliptic curves over local fields

Keywords:

rational function; height; abelian variety; dynamical system; periodic point