Farkas, Walter; **Leopold, Hans-Gerd**
Characterisations of function spaces of generalised smoothness. (English) Zbl 1116.46024
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This rather extensive paper presents a theory of function spaces of generalized smoothness which extends the theory of the well-known Besov spaces $B^s_{p,q}$ and Triebel-Lizorkin spaces $F^s_{p,q}$. The spaces are defined on the basis of a refinement of the standard decomposition method in which the following two types of sequences play a role. A sequence $N = (N_j)_{j\in\mathbb{N}_0}$ is strongly increasing if there is a positive constant d_0 and a natural number κ_0 such that $d_0 N_j \le N_k$ for all j, k with $0 \le j \le k$, and $2N_j \le N_k$ for all j, k with $j + \kappa_0 \le k$. The sequence N is of bounded growth if there exists a constant $d_1 > 0$ and $J \in \mathbb{N}_0$ such that $N_{j+1} \le d_1 N_j$ for any $j \ge J_0$. A sequence $\sigma = (\sigma_j)$ of positive real numbers is admissible if both (σ_j) and (σ_j^{-1}) are of bounded growth, i.e. $d_0\sigma_j \le \sigma_{j+1} \le d_1\sigma_j$ for all $j \in \mathbb{N}$.

To define the corresponding decomposition set $\Omega_j^{N,J} = \{\xi \in \mathbb{R}^n : |\xi| \le N_{j+J\kappa_0}\}$ if $j = 0, 1, \dots, J\kappa_0 - 1$ and $\Omega_j^{N,J} = \{\xi \in \mathbb{R}^n : N_{j-J\kappa_0} \le |\xi| \le N_{j+J\kappa_0}\}$ if $j \ge J\kappa_0$ and let $\Phi^{N,J}$ be a class of all function systems $\varphi^{N,J} = (\varphi_j^{N,J})$ be a sequence of non-negative smooth functions with a support in $\Omega_j^{N,J}$ such that $|D^\gamma \varphi_j^{N,J}(\xi)| \le c_\gamma \langle\xi\rangle^{-\gamma}$ for any $\gamma \in \mathbb{N}_0^n$, $j \in \mathbb{N}_0$, and $0 < \sum_{j=0}^\infty \varphi_j^{N,J}(\xi) = c_\varphi < \infty$ for any $\xi \in \mathbb{R}^n$. Let $1 < p < \infty$. For $1 \le q \le \infty$ the Besov space of generalized smoothness $B^{\sigma,N}_{p,q}$ is the class of tempered distributions f with the norm $\|f \mid B^{\sigma,N}_{p,q}\| = \|(\sigma_j\varphi_j^{N,J}(D)f)_{j\in\mathbb{N}_0} \mid l_q(L_p)\| < \infty$. For $1 < q < \infty$ the Triebel-Lizorkin space of generalized smoothness $F^{\sigma,N}_{p,q}$ is the class of tempered distributions f with the norm

$$\|f \mid F^{\sigma,N}_{p,q}\| = \|(\sigma_j\varphi_j^{N,J}(D)f(\cdot))_{j\in\mathbb{N}_0} \mid L_p(l_q)\| < \infty.$$

If $N_j = 2^j$ and $\sigma_j = 2^{js}$ then the spaces $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ consider with the usual Besov spaces $B^s_{p,q}$ and Triebel-Lizorkin spaces $F^s_{p,q}$. The authors investigate many properties of the spaces of generalized smoothness. They use the Michlin-Hörmander-type theorem on Fourier multipliers to show the consistency of the definition and prove the theorem of Littlewood-Paley type, $F^{1,N}_{p,2} = L_p$. They prove the analogues of the usual embeddings,

$$L_p \hookrightarrow B^{1,N}_{p,\infty} \hookrightarrow B^0_{p,\infty}, \quad B^0_{p,1} \hookrightarrow B^{1,N}_{p,1} \hookrightarrow L_p,$$

characterize the dual spaces etc.

The main results of the paper concerns the characterization with local means and the atomic decomposition under the assumption that N satisfies $\lambda_0 N_j \le N_{j+1} \le \lambda_1 N_{N+1}$ with $1 < \lambda_0 \le \lambda_1$.

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References:

[1] Adams, D.R., Hedberg, L.I.: Function spaces and potential theory. Berlin: Springer 1996 · Zbl 0834.46021
[2] Bui, Stud. Math., 119, 219 (1996)
[3] Bui, H.-Q., Paluszyński, M., Taibleson, M.: Characterization of the Besov-Lipschitz and Triebel-Lizorkin spaces. The case q<1. J. Fourier Anal. Appl. 3, 837-846 (1997) · Zbl 0897.42010
[4] Burenkov, Oper. Theory Adv. Appl., 109, 187 (1999)
[5] Bricchi, M.: Tailored function spaces and related h-sets. PhD Thesis. University of Jena 2002 · Zbl 1215.46022

[6] Cobos, F., Fernandez, D.L.: Hardy-Sobolev spaces and Besov spaces with a function parameter. Proc. Lund Conf. 1986, Lect. Notes Math. 1302, 158-170. Berlin: Springer 1986

[7] Courrège, Ph.: Sur la forme intégro-différentielle des opérateurs de C_K^⊠ dans ⊠ satisfaisant au principe du maximum. Exposé 2. Sém. Théorie du Potentiel, 1965/66, 38 pp

[8] Edmunds, Diss. Math., 380, 1 (1999)

[9] Edmunds, J. Funct. Anal., 146, 116 (1997) · Zbl 0934.46036

[10] Edmunds, D.E., Triebel, H.: Function spaces, entropy numbers, differential operators. Cambridge: Cambridge University Press 1996 · Zbl 0865.46020

[11] Edmunds, C. R. Acad. Sci. Paris, 326, 1269 (1998)

[12] Edmunds, Oper. Theory Adv. Appl., 110, 81 (1999)

[13] Farkas, Forum Math., 13, 59 (2001)

[14] Farkas, Diss. Math., 393, 1 (2001)

[15] Farkas, Jenaer Schriften zur Mathematik und Informatik, 23, 1 (2001)

[16] Fefferman, Am. J. Math., 93, 107 (1971)

[17] Frazier, Indiana Univ. Math. J., 34, 777 (1985) · Zbl 0551.46018

[18] Frazier, J. Funct. Anal., 93, 34 (1990) · Zbl 0716.46031

[19] Frazier, M., Jawerth, B.: Littlewood-Paley theory and the study of function spaces. CBMS Reg. Conf. Ser. Math. 79. Providence, RI: Am. Math. Soc. 1991 · Zbl 0757.42006

[20] Fukushima, Trans. Am. Math. Soc., 162, 185 (1971)

[21] Goldman, Dokl. Akad. Nauk SSSR, 231, 525 (1976)

[22] Goldman, Inst. Steklov, 150, 99 (1979)

[23] Goldman, Trudy Mat. Inst. Steklov, 156, 47 (1980)

[24] Goldman, Trudy Mat. Inst. Steklov, 170, 86 (1984)

[25] Goldman, Dokl. Akad. Nauk SSSR, 277, 20 (1984)

[26] Goldman, Trudy Mat. Inst. Steklov, 172, 128 (1985)

[27] Goldman, Trudy Mat. Inst. Steklov, 173, 90 (1986)

[28] Goldman, Trudy Mat. Inst. Steklov, 187, 69 (1989)

[29] Goldman, Trudy Mat. Inst. Steklov, 201, 186 (1992)

[30] Jacob, N.: Pseudo-differential operators and Markov processes. Berlin: Akademie Verlag 1996 · Zbl 0860.60002

[31] Jacob, Potential Analysis, 8, 61 (1998) · Zbl 0908.60041

[32] Jacob, N.: Pseudo-differential operators and Markov processes. Vol. 1: Fourier analysis and semigroups, World Scientific, Imperial College Press 2001 · Zbl 0987.60003

[33] Jacob, Math. Nachr., 178, 199 (1996)

[34] Kalyabin, Mat. Sb. Nov. Ser., 104, 42 (1977)

[35] Kalyabin, Dokl. Akad. Nauk SSSR, 232, 1245 (1977)

[36] Kalyabin, Izv. Akad. Nauk SSSR Ser. Mat., 42, 305 (1978)

[37] Kalyabin, Trudy Mat. Inst. Steklov, 150, 160 (1979)

[38] Kalyabin, Trudy Mat. Inst. Steklov, 156, 82 (1980)

[39] Kalyabin, Mat. Zametki, 30, 517 (1981)

[40] Kalyabin, Trudy Mat. Inst. Steklov, 181, 95 (1988)

[41] Kalyabin, Math. Nachr., 133, 7 (1987)

[42] Kudryavtsev, L.D., Nikol'skii, S.M.: Spaces of differentiable functions of several variables and imbedding theorems. Analysis III, Spaces of differentiable functions, Encyclopadia of Math. Sciences 26, 4-140. Heidelberg: Springer 1990

[43] Leopold, H.G.: Spaces of variable and generalised smoothness. Manuscript, Jena, 1990

[44] Leopold, H.G.: Embeddings and entropy numbers in Besov spaces of generalized smoothness. in: Function Spaces: The fifth conference, Lecture Notes Pure Appl. Math. 213, 323-336. Hudzik, H., Skrzypczak, L.(eds.). Marcel Dekker 2000 · Zbl 0966.46018

[45] Merucci, C.: Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces. Proc. Lund Conf. 1983. Lect. Notes Math. 1070, 183-201. Berlin: Springer · Zbl 0546.46061

[46] Moura, S.: Some properties of the spaces F^(s,Ψ)_pq(⊠^n) and B^(s,Ψ)_pq(⊠^n). Preprint 99-09, Univ. Coimbra 1999

[47] Moura, Diss. Math., 398, 1 (2001)

[48] Netrusov, Dokl. Akad. Nauk SSSR, 298, 1326 (1988)

[49] Netrusov, Trudy Mat. Inst. Steklov, 190, 159 (1989)

[50] Opic, Anal. Math., 26, 299 (2000) · Zbl 0973.46021

[51] Peetre, Ark. Mat., 13, 123 (1975)

[52] Rychkov, Trudy Mat. Inst. Steklov, 227, 286 (1999)

[53] Schilling, R.L.: Zum Pfadverhalten von Markovschen Prozessen, die mit Lévy Prozessen vergleichbar sind. Dissertation Universität Erlangen-Nürnberg, Erlangen 1994 · Zbl 0805.60032

[54] Schilling, J. Aust. Math. Soc., 64, 368 (1998)

[55] Schilling, Potential Anal., 9, 91 (1998) · Zbl 0917.60066

[56] Schmeisser, H.J., Triebel, H.: Topics in Fourier analysis and function spaces. Leipzig: Geest \& Portig 1987 · Zbl 0661.46024

[57] Stein, E.M., Weiss, G.: Introduction to Fourier analysis on euclidean spaces. Princeton, NJ: Princeton Univ. Press 1971 · Zbl 0232.42007

[58] Strömberg, J.O., Torchinsky, A.: Weighted Hardy spaces. Lect. Notes Math. 1381. Berlin: Springer 1989 · Zbl 0676.42021

[59] Triebel, H.: Fourier analysis and function spaces. Teubner-Texte Math. 7. Leipzig: Teubner 1977 · Zbl 0345.42003

[60] Triebel, H.: Interpolation theory, function spaces, differential operators. Amsterdam: North Holland 1978 · Zbl 0387.46033

[61] Triebel, H.: Theory of function spaces. Leipzig: Geest \& Portig and Basel: Birkhäuser 1983

[62] Triebel, H.: Theory of function spaces. Russian ed. Moscow: Mir 1986 · Zbl 0611.46036

[63] Triebel, J. Approximation Theory, 52, 162 (1988) · Zbl 0644.46017

[64] Triebel, H.: Theory of function spaces II. Basel: Birkhäuser 1992 · Zbl 0763.46025

[65] Triebel, H.: Fractals and spectra, related to Fourier Analysis and Function Spaces. Basel: Birkhäuser 1997 · Zbl 0898.46030

[66] Triebel, H.: The structure of functions. Basel: Birkhäuser 2001 · Zbl 0984.46021

[67] Triebel, Stud. Math., 121, 149 (1996)