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A fractional generalization of the Poisson processes. (English) Zbl 1087.60064

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The concept of renewal process has been developed as a stochastic model for describing the class of counting processes for which the times between successive events (waiting times) are independent and identically distributed, non-negative random variables, obeying a given probability law. For a renewal process having waiting times T_1, T_2,..., let $t_0 = 0$, $t_k = \sum_{j=1}^{k} T_j$, $k \geq 1$ (t_k is the time of the kth renewal). The process is specified if we know the probability law for the waiting times ($\phi(t)$ is its density function and $\Phi(t)$ the distribution function of the waiting times). $\Psi(t) = 1 - \Phi(t)$ is the survival probability. A relevant quantity is the counting function $N(t) = \max\{k : t_k \leq t,\ k = 0, 1, 2, \dots\}$. In particular, $\Psi(t) = P(N(t) = 0)$. The most celebrated renewal process is the Poisson process characterized by a waiting time density function of exponential type. The survival probability for the Poisson renewal process obeys the ordinary differential equation of relaxation type: $d\Psi(t)/dt = -\Psi(t)$, $t \geq 0$ and $\Psi(0^+) = 0$. A "fractional" generalization of the Poisson renewal process is simply obtained by generalizing the differential equation $d\Psi(t)/dt = -\Psi(t)$ replacing there the first derivative with the integro-differential operator ${}_tD_*^\beta$ that is interpreted as the fractional derivative of order β, $0 < \beta \leq 1$. In the present paper the authors analyze a non-Markovian renewal process with a waiting time distribution described by the Mittag-Leffler function. The Mittag-Leffler function of parameter β is defined in the complex plane $z \in \mathcal{C}$ by the power series: $E_\beta(z) = \sum_{n=0}^{\infty} z^n/\Gamma(\beta\, n + 1)$. It turns out to be an entire function of order β which reduces for $\beta = 1$ to $\exp(z)$. The solution of the equation ${}_tD_*^\beta \Psi(t) = -\Psi(t)$ is known to be $\Psi(t) = E_\beta(-t^\beta)$, $t \geq 0$ and $0 < \beta \leq 1$, see for example *R. Gorenflo* and *F. Mainardi* [in: Fractals and fractional calculus in continuum mechanics, 223–276, (1997)]. In contrast to Poisson case $\beta = 1$, in the case $0 < \beta < 1$ for large t the functions $\Psi(t)$ and $\phi(t)$ no longer decay exponentially but algebraically. As a consequence of the power-law asymptotics the process turns to be no longer Markovian but of long-memory type. However, for $0 < \beta < 1$ both functions $\Psi(t)$ and $\phi(t)$ keep the "completely monotonic" character of the Poisson case. Completely monotonicity of the function F means $(-1)^n\, d^n F(t)/dt^n \geq 0$ for $n = 0, 1, 2, \dots$ and $t \geq 0$ or equivalently, representability of $F(t)$ as real Laplace transform of nonnegative generalized functions (or measures), see Gorenflo and Mainardi [loc. cit.].

The paper is organized as follows: Section 1 recalls some notions from renewal processes. Section 2 considers the classical case, the Poisson renewal process. Section 3 deals with the Mittag-Leffler generalization of the Poisson renewal process and asymptotics for functions $\Psi(t)$ and $\phi(t)$ for $t \to \infty$. In Section 4 the authors have obtained that the Mittag-Leffler probability distribution is the limiting distribution for the thinning procedure of a generic renewal process with waiting time density of power law character. Section 5 deals with renewal processes by reward, that means that every renewal instant as space-like variable makes a random jump from its previous position to a new point in space. The stochastic evolution of the space variable in time is modelled by an integro-differential equation which, by containing a time fractional derivative, can be considered as the fractional generalization of the classical Kolmogorov-Feller equation of the compound Poisson process.

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MSC:

60K05	Renewal theory
60G55	Point processes (e.g., Poisson, Cox, Hawkes processes)
30D10	Representations of entire functions of one complex variable by series and integrals

Keywords:

Renewal processes; Mittag-Leffler function; thinning procedure.

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