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Multidimensional diffusion processes. (English) Zbl 0426.60069
Grundlehren der mathematischen Wissenschaften. 233. Berlin, Heidelberg, New York: Springer-Verlag. XII, 338 p. DM 70.00; $ 38.50 (1979).

This book develops the martingale approach to Markov diffusion processes, along the lines initiated by the same authors in two papers that appeared in 1969. This approach has been since widely used to construct and handle various types of Markov processes. The authors have chosen to cover the case of diffusion processes in $\mathbb{R}^d$, in a very complete way. Other aspects of the same approach, as well as a complementary bibliography, can be found in [*J. Jacod*, Calcul stochastique et problèmes de martingales. Lecture Notes in Mathematics. 714. Berlin-Heidelberg-New York: Springer-Verlag (1979; Zbl 0414.60053)]. The martingale approach to diffusion processes consists in studying the following "martingale problem": Given functions $a : [0,\infty) \times \mathbb{R}^d \to S^d$ (the set of symmetric ≥ 0 $d \times d$-matrices) and $b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$, define the operator:

$$L_u = \frac{1}{2}\sum_{i,j} a_{i,j}(u,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(u,x)\frac{\partial}{\partial x_i}$$

A solution to the corresponding martingale problem, starting from (s,x), is a probability measure P on the space of continuous paths $C(\mathbb{R}_+;\mathbb{R}^d)$, such that (i) $P(x(t) = x, 0 \leq t \leq s) = 1$; (ii) $f(x(t)) - \int_s^t L_u f(x(u))\,du$ is a P-martingale for all $f \in C_0^\infty(\mathbb{R}^d)$.

The following questions are studied by the authors: (i) existence of a solution; (ii) uniqueness; (iii) additional properties, in the case one has both existence and uniqueness. The first third of the book is concerned with other ways of constructing a diffusion process, which provide in some cases a solution to the martingale problem: the method of Kolmogorov backward equation, and the method of Ito stochastic differential equation. Then, the martingale problem is studied, starting with existence. Uniqueness, which is harder to prove and requires the non-degeneracy of the diffusion coefficient a, is studied in detail. The Feller property of the process is proved under similar conditions. Also, the Cameron-Martin-Girsanov formula is established. A chapter is devoted to the results by Yamada and Watanabe an Ito's uniqueness, and its implication for the uniqueness of a solution to the martingale problem. The last chapter is concerned with the non-unique case, where a "good" selection can be made among the solutions. Whereas the theory is first developed in the case of bounded coefficients, one of the last chapters is devoted to its extension to the case of locally bounded coefficients. Basically, the results are the same provided the process does not explode. Conditions for explosion and non explosion are given.

The next-to-last chapter is concerned with limit theorems: convergence in law of diffusion processes to a diffusion process, and convergence of Markov chains to a diffusion process (result called invariance theorem). Here, the martingale formulation proves to be very powerful since martingale methods are very useful in proving limit theorems. Some of these are part of the preliminary material in probability, which forms Chapter 1. An appendix at the end of the book establishes some estimates from the theory of singular integrals. A good part of the theory presented in the book does need these results from analysis.

This book gives a very complete treatment of the theory of diffusion processes in $\mathbb{R}^d$. The various connections with partial differential equations are explained. The text itself is complemented with a great number of exercises, for which the main ideas of proof are often indicated. The book is very carefully written. Many introductions and comments explain the ideas, which might unless be hidden behind the often very intricate technicalities. The mathematical rigour is kept at a very high level. For instance, the construction of Ito's stochastic integral is given without neither completing the σ-algebras, nor skipping the difficulty about anticipative null-measure sets. This book constitutes a very complete – and for some results unique – reference text on diffusion processes. It should prove to be very useful to both theoretical and applied probabilists. For instance, the limit theorems are very useful in the theories of stochastic control and filtering. Though all the results needed from the theory of Markov processes, martingales, weak convergence of probability measures are proved in Chapter 1, the reader should have a good background in probability theory as well as in analysis, with some elementary functional analysis. I am sure that this book will be used both by researchers and teachers, and will become a commonly used reference book among scientists interested in the theory of diffusion processes.

Reviewer: Etienne Pardoux (Marseille)

For a scan of this review see the web version.

MSC:

60J60	Diffusion processes
60G44	Martingales with continuous parameter
60H10	Stochastic ordinary differential equations (aspects of stochastic analysis)
60F99	Limit theorems in probability theory
93Exx	Stochastic systems and control

Keywords:

multidimensional diffusion processes; martingale approach; stochastic differential equation; limit theorems; Cameron-Martin-Girsanov formula; stochastic control and filtering