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The boundary of the Eisenstein symbol. (English) Zbl 0729.11027

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For an elliptic curve E over a field F (supposed to have a nontrivial discrete valuation v, valuation ring $\mathcal{O}$ and perfect residue field k) with a finite subgroup scheme $P \subset E$ defined over F, and for any integer $n \geq 1$, one has the Eisenstein symbol map

$$\mathcal{E}_P^n : \mathbb{Q}[P]^0 \to H_{\mathcal{M}}^{n+1}(E^n, \mathbb{Q}(n+1))_{\mathrm{sgn}},$$

where $H^i_{\mathcal{M}}(-, \mathbb{Q}(j))$ is motivic cohomology, $\mathbb{Q}[P]^0$ is the $\mathbb{Q}$-vector space of $\mathrm{Gal}(\bar{F}/F)$-invariant functions $\beta : P(\bar{F}) \to \mathbb{Q}$ satisfying $\sum_{x\in P(\bar{F})} \beta(x) = 0$, E^n is identified with the kernel of the sum map $E^{n+1} \to E$ (thus giving an action of the symmetric group $\mathcal{S}_{n+1}$ on E^n), and where the subscript 'sgn' denotes the image under the projector

$$\prod_{\mathrm{sgn}} = \frac{1}{(n+1)!} \sum_{\sigma\in\mathcal{S}_{n+1}} \mathrm{sgn}(\sigma) \cdot \sigma.$$

Write E/k for the special fibre of the minimal regular model $E/\mathcal{O}$ of E and suppose that E/k is a Néron N-gon for some $N \geq 1$. Furthermore suppose that P extends to a finite flat subgroup scheme $P/\mathcal{O}$ of the Néron model of E over $\mathcal{O}$.

Also, let $\overset{\circ}{E}$ denote the connected component of the Néron model of E over $\mathcal{O}$. An isomorphism $\overset{\circ}{E}/k \xrightarrow{\sim} \mathbb{G}_m$ induces a bijection between $\mathbb{Z}/N\mathbb{Z}$ and the set of components C_ν of E/k. Thus $E/k = \cup_{\nu\in\mathbb{Z}/N\mathbb{Z}} C_\nu$. For $\beta \in \mathbb{Q}[P]^0$ let $d_\beta(\nu)$ be the degree of the restriction of the flat extension of β to C_ν. The localization sequence for the pair $(\overset{\circ}{E}{}^n/\mathcal{O}, \overset{\circ}{E}{}^n/k)$ gives a boundary map

$$\partial^n : H_{\mathcal{M}}^{n+1}(E^n, \mathbb{Q}(n+1))_{\mathrm{sgn}} \to H^n_{\mathcal{M}}(\overset{\circ}{E}{}^n/k, \mathbb{Q}(n))_{\mathrm{sgn}}.$$

The target space is a 1-dimensional $\mathbb{Q}$-vector space generated by an element of the form $\Phi^n_n = \prod_{\mathrm{sgn}}(y_0 \cup \ldots \cup y_n)$, where $y_0y_1\ldots y_n = 1$, and y_i, $1 \leq i \leq n$, is a coordinate on the ith copy of $\mathbb{G}_{m/k}$. The main result of the paper is following theorem:

$$\partial^n \circ \mathcal{E}^n_P(\beta) = C^n_{P,N} \left(\sum_{\nu\in\mathbb{Z}/N\mathbb{Z}} d_\beta(\nu) B_{n+2}\left(\langle \frac{\nu}{N} \rangle\right)\right) \cdot \Phi^n_n,$$

where $C^n_{P,N}$ is an explicit nonzero constant, $B_k(X)$ is the kth Bernoulli polynomial, and $0 \leq \langle x\rangle < 1$ is a representative of $x \in \mathbb{Q}/\mathbb{Z}$.

For the proof one may restrict to the situation where E/k is an untwisted Néron N-gon with $N \geq 3$, $P = \mu_n \times \mathbb{Z}/N\mathbb{Z} \subset E(F)$ is a level N structure on E, and P/k gives the standard level N structure on $(E/k)^{\mathrm{smooth}} = \mathbb{G}_m \times \mathbb{Z}/N\mathbb{Z}$. Then $C^n_{P,N}$ turns out to be $\pm N^n(n+1)/(n+2)!$.

The theorem is shown to follow from an explicit formula for the boundary map

$$\partial^n_v : H^{n+1}_{\mathcal{M}}(U^{n'}/F, \mathbb{Q}(n+1))^{P^n}_{\mathrm{sgn}} \to H^n_{\mathcal{M}}(U^{n'}/k, \mathbb{Q}(n))^{P^n}_{\mathrm{sgn}},$$

where $H^\bullet_{\mathcal{M}}(U^{n'}, \mathbb{Q}(*))^{P^n}_{\mathrm{sgn}}$ is a suitable $P(\bar{F})^n$-invariant sgn-part of the motivic cohomology of $U^{n'} = \{(x_1, \ldots, x_n) \in E^n | x_i \notin P, \forall i, 0 \leq i \leq n\} \subset E^n$, with $x_0 = -x_1 - \ldots - x_n$. One defines a map

$$\Theta^n_P : \mathbb{Q}[P]^{0\otimes(n+1)} \to H^{n+1}_{\mathcal{M}}(U^{n'}, \mathbb{Q}(n+1))^{P^n}_{\mathrm{sgn}}$$

and then the formula for $\partial_\nu \circ \Theta^n_P(\otimes\beta_i)$ involves, among other things, a sum of expressions containing $\zeta \in \mu N$, $\zeta \neq 1$, and this leads, on account of their distributional property, to the Bernoulli polynomials. The explicit calculation uses the fact that the boundary maps in Milnor and Quillen K-theory agree.

Then the theorem is verified for the case $n = 1$ and F a number field.

The general case consists in the "weight decomposition" of $H^\bullet_{\mathcal{M}}(U^{n'}/F, \mathbb{Q}(*))^{P^n}_{\text{sgn}}$ under the "L^{-1}"-multiplication. Actually, this "L^{-1}"-multiplication ($L \geq 1$ an integer) induces a Galois covering $[\times L] : \tilde{U}^{n'} \to U^{n'}$ and a homomorphism on (motivic) cohomology that plays a role throughout. The main step is a result, due to Beilinson and Deninger, which identifies $H^\bullet_{\mathcal{M}}(E^n, \mathbb{Q}(*))_{\text{sgn}}$ with the L^{-n}-eigenspace (for a certain endomorphism) of $H^\bullet_{\mathcal{M}}(U^{n'}, \mathbb{Q}(*))^{P^n}_{\text{sgn}}$. The Eisenstein symbol $\mathcal{E}^n_P(\beta)$ is then defined as the projection of $\Theta^n_P(\beta \otimes \alpha^{\otimes n})$, $\alpha = \sum_{x \in P(\bar{F})} (0) - (x)$, into the L^{-n}-eigenspace, viewed as an element of $H^{n+1}_{\mathcal{M}}(E^n, \mathbb{Q}(n+1))$. If F is a number field and v is a place of bad reduction of E one obtains a description of the 'integral' cohomology
$$H^{n+1}_{\mathcal{M}}(E^n/F, \mathbb{Q}(n+1))_{\mathbb{Z}} \subset H^{n+1}_{\mathcal{M}}(E^n/F, \mathbb{Q}(n+1)).$$
Also, in the modular case, one obtains a new proof of a result of Beilinson which says that the boundary map
$$\partial : H^{n+1}_{\mathcal{M}}(E^n, \mathbb{Q}(n+1))_{\text{sgn}} \to \{f : \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Q} \mid f(g \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}) = f(g) = (-1)^n f(-g)\},$$
where E is the universal elliptic curve with level N structure, defined over the function field of the modular curve of level N, $N \geq 3$, is an isomorphism on the image of the Eisenstein symbol.

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MSC:

11G05 Elliptic curves over global fields
11G40 L-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture
14C35 Applications of methods of algebraic K-theory in algebraic geometry
11F67 Special values of automorphic L-series, periods of automorphic forms, cohomology, modular symbols
19F15 Symbols and arithmetic (K-theoretic aspects)
19D45 Higher symbols, Milnor K-theory

Keywords:

Eisenstein symbol map; motivic cohomology; Néron model; Bernoulli polynomials; boundary maps; K-theory; place of bad reduction; elliptic curve; modular curve

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References:

[1] Beilinson, A.A.: Higher regulators and values ofL-functions. J. Sov. Math.30, 2036-2070 (1985) · Zbl 0588.14013 · doi:10.1007/BF02105861

[2] Beilinson, A.A.: Higher regulators of modular curves. Contemp. Math.55, 1-34 (1986) · Zbl 0609.14006

[3] Bloch, S.: Lectures on algebraic cycles. Duke Univ. Math. Series IV (1980) · Zbl 0436.14003

[4] Bloch, S., Grayson, D.:K 2 and theL-functions of elliptic curves. Computer calculations. Contemp. Math.55, 79-88 (1986) · Zbl 0629.14002

[5] Deligne, P.: Formes modulaires et repr?sentationsl-adiques. S?minaire Bourbaki 1968/69, exp. 355 (Lect. Notes Math., vol. 179, pp. 139-172) Berlin Heidelberg New York: Springer 1968

[6] Deligne, P.: Letter to C. Soul?, 20 Jan. 1985 (unpublished)

[7] Deninger, C.: Higher regulators and HeckeL-series of imaginary quadratic fields I. Invent. Math.96, 1-69 (1989) · Zbl 0721.14004 · doi:10.1007/BF01393970

[8] Deninger, C., Scholl, A.J.: The Beilinson conjectures. In:L-functions and arithmetic, Coates, J.H., Taylor, M.J. (eds), pp. 173-209 Cambridge: Cambridge Univ. Press 1991 · Zbl 0729.14002

[9] Grayson, D.: HigherK-theory: II, after D. Quillen. In: AlgebraicK-theory. Evanston Proc., Stein, M.R. (ed.) ((Lect. Notes Math., vol. 551, pp. 217-240). Berlin Heidelberg New York: Springer 1976

[10] Loday, J.L.:K-th?orie alg?brique et repr?sentations de groupes. Ann Sci. Ec. Norm. Super. IV S?r.9, 309-377 (1976) · Zbl 0362.18014

[11] Mestre, J.-F., Schappacher, N.: S?ries de Kronecker et fonctionsL des puissances sym?triques de courbes elliptiques surQ. In: Arithmetic algebraic geometry, Van de Geer, G., Oort, F., Steenbrink, J. (eds.) (Prog. Math., vol 89, pp. 209-245. Boston Basel Berlin: Birkh?user 1991

[12] Ramakrishnan, D.: Regulators, algebraic cycles, and values ofL-functions. Contemp. Math.83, 183-310 (1989)

[13] Scholl, A.J.: Motives for modular forms. Invent. math.100, 419-430 (1990_ · Zbl 0760.14002 · doi:10.1007/BF01231194

[14] Schneider, P.: Introduction to the Beilinson conjectures. In: Beilinson's Conjectures on special values ofL-functions. Rapoport, M., Schappacher, N., Schneider, P. (eds.) Perspecives in Math.4, 1-35. New York: Academic Press 1988

[15] Soul?, C.: Groupes de Chow etK-th?orie de vari?t?s sur un corps fini. Math. Ann.268, 317-345 (1984) · Zbl 0573.14001 · doi:10.1007/BF01457062