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The analysis of linear partial differential operators. III: Pseudo-differential operators. (English) Zbl 0601.35001

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[For the review of Vols. I, II see Zbl 0521.35001 and Zbl 0521.35002.]

Whereas the previous volumes I and II were more or less organized similar to the author's first treatise on partial differential equations and dealt with the analysis of distributions and constant coefficient operators (with many extensions and improvements, of course), volumes III and IV contain almost entirely techniques and results developed or completed after 1965. To most topics covered the author has made himself substantial or decisive contributions. Naturally, the work of others is incorporated as well but often in simplified or extended form. Very impressive features of this treatise are the completeness of the results given, the wealth of material, and the marvellous elegancy and homogeneity of presentation. Each chapter has a very helpful short summary and historical notes which outline the underlying ideas and their development, and also explain the connection of the material with other parts of the work. We now summarize the contents of volume III.

Chapter XVII is devoted to second order elliptic operators since in this case there are either simpler proofs or more refined results than for general elliptic operators. § 17.1 gives interior regularity and local existence results based on a constant coefficient approximation and the Neumann series. § 17.2 deals with unique continuation theorems starting with a result for operators of arbitrary order (Theorem 17.2.1) then giving a very general result in the second order case (Theorem 17.2.6). This is applied to global existence (Theorem 17.2.7) and absence of embedded eigenvalues (Theorem 17.2.8). § 17.3 gives existence and regularity results for the Dirichlet problem using a constant coefficient parametrix and a reflection argument (Theorems 17.3.1 and 17.3.2), and a short but enlightening discussion of other classical methods. § 17.4 introduces the Hadamard parametrix construction which because of its precision has become such a powerful tool in the analysis of second order elliptic equations. It is constructed for the resolvent and the wave equation emphasizing the importance of geodesic coordinates; the important application to the heat kernel is not explicitly mentioned but it follows from the resolvent by integration. Then the construction is extended to the mixed problem for the wave equation with Dirichlet boundary conditions via the reflected exponential map (Proposition 17.4.4) which becomes important in § 17.5. Here the sharp asymptotics of the spectral function and the eigenvalues for the Dirichlet problem are derived (Theorems 17.5.9, 17.5.10, Corollary 17.5.11).

This section (and its extension in Chapter XXIX on spectral asymptotics) is one of the highlights of this treatise. It presents the answers to longstanding questions in eigenvalue distribution, in particular on manifolds with boundary, and illustrates the impressive progress that has been made since the author's first book on partial differential equations appeared where eigenvalue problems were not even mentioned.

Chapter XVIII deals with pseudodifferential operators (ψdo's) and their basic properties. Section 18.1 develops the basic calculus [asymptotic series, composition, adjoints, invariance, L^2- boundedness] including an elementary proof of the sharp Gårding inequality (Theorem 18.1.14) and the discussion of ψ do's on manifolds, acting on half-densities rather than on functions. Section 18.2 deals with the spaces $I^m(X, Y; E)$ of distributional sections of a C^∞ vector bundle E over a manifold X, conormal with respect to a submanifold Y (Definition 18.2.6). The mapping properties of ψ do's (Theorem 18.2.7) and the local description in terms of oscillatory integrals are given (Theorem 18.2.8), then the notion of principal symbol is developed (Theorem 18.2.11) and related to the action of ψdo's (Theorem 18.2.12). As a natural application then appears a necessary and sufficient condition for a ψ do to have the transmission property (Definition 18.2.13 and Theorem 18.2.15), based on the important fact that restrictions of smooth functions to a halfspace are conormal with respect to the boundary. Theorem 18.2.17 describes in detail the boundary behavior of ψdo's with the transmission property.

Section 18.3 presents the global calculus of totally characteristic operators due to Melrose. These are ψdo's on manifolds with boundary including maps $C_0^\infty(\partial X) \to C(\partial X)$, built up from tangential first order differential operators. They are characterized globally by their kernels which are push forwards of certain conormal distributions on the "stretched product" $X\hat{\times}X$ (Definition 18.3.17) and the calculus is quite analogous to the calculus of ψdo's on manifolds without boundary through locally more complicated (Theorems 18.3.5, 18.3.8, 18.3.11). Totally characteristic operators are also naturally encountered in the analysis of conic singularities. The purpose of Section 18.4 is to extend the estimates for the Gauss transform crucial in the development of the calculus of ψdo's (Theorem 18.1.7) to larger classes of symbols $S(m,g)$ for example the classes $S^m_{\rho,\delta}$ and those motivated by the study of local solvability for operators of principal type (Theorems 18.4.10, 18.4.11).

Section 18.5 introduces the Weyl calculus of symbols associating with $a\in S^m_{\rho,\delta}$ and operator $a^\omega(x,D)$ such that the adjoint of a^ω is $\bar{a}^\omega$. The composition formula is obtained in great generality using the results on the Gauss transform from Section 18.4 (Theorems 18.5.4, 18.5.5). The Gauss transform also relates the Weyl calculus to the calculus of ψdo's thereby generalizing considerably the composition result Theorem 18.1.8. Two other important features of the Weyl calculus are its symplectic invariance (Theorem 18.5.9) and the natural occurrence of the subprincipal symbol. Section 18.6 gives boundedness results for the operators a^ω with $a\in S(m,g)$ for suitable m,g in S,S^1 (Theorem 18.6.2) and L^2 (Theorem 18.6.3), and a necessary and sufficient condition for compactness (Theorem 18.6.6). Then the sharp Gårding inequality of Fefferman and Phong is proved (Theorem 18.6.8) together with a vector-valued analogue (Theorem 18.6.14).

Chapter XIX is devoted to Fredholm properties of elliptic operators on compact manifolds without boundary. Section 19.1 reviews the basic properties of Fredholm operators in Banach spaces. Besides the standard material one finds a useful refined stability theorem (Theorem 19.1.10), a trace formula for the index (Proposition 19.1.14), and results on the invariance of Euler numbers in passing to homology (Theorem 19.1.15, Theorem 19.1.20). Sections 19.2 and 19.3 together provide a technique to calculate the index of an elliptic ψdo. It is shown that elliptic ψdo's are Fredholm operators as operators on Sobolev spaces of arbitrary order (Theorem 19.2.1), that the index depends on the principal symbol only (Theorem 19.2.2), and that it is enough to consider operators of order 0 (Theorem 19.2.3) (actually a slightly larger class of operators, cf. Theorems 19.2.4, 19.2.5). Regarding a compact manifold X as a submanifold of some $\mathbb{R}^\nu$ the product construction (Theorem 19.2.5) applied to the given operator on X and the Bott operator (Theorem 19.2.12) acting on the normal bundle of X yields an elliptic operator in $\mathbb{R}^\nu$ with the same index as P (Theorem 19.2.13). This operator is deformed further to be trivial at infinity, and for such operators the index can be calculated explicitly (Theorem 19.3.1); here again the Weyl calculus proves to be very useful. The explicit cohomological formula of Atiyah and Singer is not derived but a proof is indicated in the notes. Section 19.4 contains an elegant version of the Lefschetz fixed point formula using wave fronts (Theorem 19.4.1) and section 19.5 exploits the relationship between Fredholmness and ellipticity and also introduces ellipticity in the sense of Douglis-Nirenberg.

Chapter XX deals with elliptic boundary value problems.They are defined for differential operators in Section 20.1 and proved to define Fredholm operators (Theorem 20.1.2) via reduction to the boundary (Theorems 20.1.3 and 20.1.7); then the stability of the index and global regularity is proved in Theorem 20.1.8. This is extended to more general operators in preparation of the index calculation (Theorem 20.1.8') and a microlocal version of regularity at the boundary is given (Theorem 20.1.4). Section 20.2 contains some preliminary results on ordinary differential equations needed in Section 20.3 where an elliptic boundary value problem is deformed to another one the index of which equals the index of an associated elliptic problem on the double of the manifold; thus the problem is in principle reduced to the calculations of Chapter IX. Section 20.5 shortly discusses reduction to the boundary for non-elliptic boundary value problems.

Chapter XXI develops the symplectic geometry naturally related to the study of pseudodifferential operators via the principal symbol and its characteristic equation; it also gives the basis for the definition of FIO's in volume IV.Section 21.1 contains the functional facts, section 21.2 gives various important results on normal forms for submanifolds of symplectic manifolds (Theorems 21.2.10, 21.2.14, 21.2.16). Section 21.3 studies the normal forms of symbols that can be achieved by symplectic transformations. Section 21.4 deals with the singular situation of maps with folds. Normal forms for one (Theorem 21.4.2) or two related folding maps (Theorem 21.4.4) are given and lead to the symplectic equivalence of all pairs of glancing hypersurfaces (Theorem 21.4.8); analogues of these results in the more complicated homogeneous case are also given. In section 21.5 one finds the symplectic classification of quadratic forms (Theorems 21.5.3, 21.5.4) which becomes important in the study of hypoelliptic equations. The manifold of all Lagrangian subspaces of a symplectic vector space is studied in Section 21.6, and the Maslov bundle is introduced as the symbol space of certain distributions naturally attached to pairs of Lagrangian subspaces.

Chapter XXII is devoted to results on hypoellipticity. Section 22.1 deals with hypoelliptic ψ do's which have a parametrix in the class $S^m_{\rho,\delta}$ (Theorem 22.1.4). The non-invariance of this class (for $\rho = 1/2$) is illustrated with the heat operator, motivating further study of hypoellipticity. The next section reveals a condition for second order equations modelled after the Kolmogorov equation (already treated explicitly in XII.6) given in Theorem 22.2.1; this is further generalized to a local result in Theorem 22.2.6. Sections 22.3 and 22.4 present fairly complete results on hypoellipticity with loss of one derivative. In the self-adjoint case this is essentially characterized by Melin's inequality (Theorem 22.3.3 and Proposition 22.4.1). The extension to principal symbols with values in a suitable complex cone (which requires considerable additional work) is given in Theorems 22.4.14, 22.4.15.

Chapter XXIII treats the strictly hyperbolic Cauchy problem.Section 23.1 proves the correctness of the Cauchy problem (Theorem 23.1.2) and the propagation of singularities (Theorem 23.1.4) for first order operators using the energy integral method. This is extended in section 23.2 to strictly hyperbolic operators of arbitrary order (Definition 23.2.3) using factorization by first order operators (theorems 23.2.4, 23.2.10). In the next section the necessity of strict hyperbolicity is examined leading to a weaker condition in Theorem 23.3.1 and a sharper version of it for operators of principal type (Proposition 23.3.3). This latter condition is satisfied e.g. by the important Tricomi operator, and this motivates the study of this sufficiency in Section 23.4 where satisfying results on correctness (Theorem 23.4.5) and regularity (Theorem 23.4.8) of the Cauchy problem are given.

Chapter XXIV deals with the mixed problem for second order operators i.e. the Dirichlet-Cauchy problem.Section 24.1 examines strictly hyperbolic operators. Exploiting the Lorentzian geometry induced by the principal symbol the crucial energy estimate is obtained leading to existence and uniqueness in this case (Theorem 24.1.1). Section 24.2 introduces the more general setting of the noncharacteristic boundary value problem for second order operators with real principal symbol, and begins the study of boundary regularity of the solutions. Locally the principal symbol p has normal form $p(x,\xi) = \xi_1^2 - r(x,\xi_2,...,\xi_n)$ and $T^*(\partial X)$ has to be divided according to $r < 0$ (elliptic set), $r > 0$ (hyperbolic set), or $r = 0$ (glancing set). Theorem 24.2.1 describes the propagation of singularities near points in the hyperbolic set. It is the microlocal version of the reflection law of optics and leads naturally to the notion of broken bicharacteristic (Definition 24.2.2). The result is illustrated by an example with singularities exactly along one broken bicharacteristics (Theorem 24.2.3).

In the next section the analysis is extended to tangential bicharacteristics. The glancing set is further divided according to the order of vanishing of $H_p^j\phi$ where H_p is the Hamilton vector field of p and ϕ is a defining function for the boundary, and the set with vanishing of order two precisely splits in the diffractive and gliding part according to $H_p^2\phi > 0$ or < 0, respectively. Example 24.3.3 illustrates the drastically different behavior of broken bicharacteristics starting in the diffractive or in the gliding set. This motivates the notion of generalized bicharacteristics (Definition 24.3.7) which include the limits of broken bicharacteristics (Proposition 24.3.12) and replace the bicharacteristic flow on a compact manifold without boundary. Theorems 24.3.8 and 24.3.9 describe the behavior of generalized bicharacteristics starting from points in the glancing set with finite order of vanishing.

Section 24.4 presents the propagation of singularities in the diffractive case (Theorem 24.4.1) based on a delicate energy estimate. This is extended in section 24.5 to the entire glancing set (Theorem 24.5.3) and the result is shown to be almost optimal (Theorem 24.5.4). Section 24.6 is motivated by the Tricomi operator and supplies a result on hypoellipticity in the intersection of the gliding and the complement of the closure of the hypoelliptic set (Theorem 24.6.2). Also, in the intersection of the diffractive and the complement of the closure of the elliptic set solutions are constructed with singularities precisely on one bicharacteristic (Theorem 24.6.5); in this case Theorem 24.5.4 is not applicable. Finally, Section 24.7 supplements the proof of Lemma 17.5.14 based on a version of Theorem 24.5.3 for operators depending on a parameter.

Reviewer: J. Brüning

MSC:

35-02 Research exposition (monographs, survey articles) pertaining to partial differential equations

35Sxx Pseudodifferential operators and other generalizations of partial differential operators

58J40 Pseudodifferential and Fourier integral operators on manifolds

47G30 Pseudodifferential operators

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