Mumford, David

Tata lectures on theta. II: Jacobian theta functions and differential equations. With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman, and H. Umemura. (English)

Zbl 0549.14014
Progress in Mathematics, Vol. 43. Boston-Basel-Stuttgart: Birkhäuser. XIV, 272 p. DM 62.00 (1984).

This is the second volume of the book, the first volume of which consists of two chapters I, II (1983; Zbl 0509.14049). Chapter I is devoted to the theta functions θ of genus one, or geometric and arithmetic theory of elliptic curves; in Chapter II θ is generalized to the case of several variables and the author develops the theory of abelian varieties, especially Jacobian varieties associated to compact Riemann surfaces mainly following Riemann's lines, with applications to modular forms.

This volume II includes three parts III a-III c, and the third part III c is written by *H. Umemura*, where he proves that any algebraic equation is solved by using hyperelliptic theta functions and hyperelliptic integrals (instead of radicals in the case of abelian polynomials). Here Thomae's formula (theorem 8.1 in III a) plays an essential role. III a and III b are concerned only with theta functions θ (z,Ω) in the case, where Ω are symmetric matrices arising as period matrices of Riemann surfaces C; C is hyperelliptic in III a, and is an arbitrary Riemann surface in III b. In both cases the author lays stress on producing solutions of some important non-linear partial differential equations from these special θ 's. Now we look over the outline of each part.

III a is the main part of this volume and a beautiful combination of the cassical geometry of hyperelliptic curves and dynamical systems, with applications to non-linear PDE's. After the reader reviews some algebraic geometric background in § 0, he learns a parameter variety of effective divisors D of degree ν, on a hyperelliptic curve C (due to Jacobi) in § 1. In this case D is not arbitrary, but this parameter variety and the coordinate corresponding to the divisor D (which we call "Jacobi coordinate" though it is not named in the paper), seems more convenient for our purpose than the Chow variety and coordinate which are more generally applicable. When C is of genus g and $\nu = g$, the parameter variety is used for algebraic construction of the Jacobian variety Jac C of C (§ 2), and the Jacobi coordinate corresponding to D is very explicitly determined later (§ 5, § 7).

Under this model of Jacobi the translation invariant vector fields on Jac C are given by explicit formulae (§ 3), and these models and formulae are used to solve the Neumann dynamical system (§ 4). § 5 links the above theory with the analytic and Riemann theory of Jac C as a complex torus $\mathbb{C}^g/L_\Omega$ in chapter II, and the Jacobi coordinates are determined (up to a scalar) as meromorphic functions on $\mathbb{C}^g/L_\Omega$ using theta functions. These sections §§ 0-5 cover good fundamental theory of hyperelliptic curves, which enables us to prove that hyperelliptic thetas have the fundamental vanishing property (corollary 6.7) and that this property characterizes hyperelliptic period matrices Ω among all matrices in the Siegel upper half space (theorem 9.1). These are the main results in § 6 and § 9.

On the other hand between these two sections, Frobenius' theta formula (§ 7) and Thomae's formula (§ 8) are given; the former formula has two applications: (1) evaluating the scalar part of the Jacobian coordinates and (2) giving explicitly via thetas the solutions of Neumann's dynamical system; the latter formula determines the affine ring of the moduli space of hyperelliptic curves and is also used in III c as we mentioned before. The application (1) in § 7 and the theory in § 4 (corollary 4.9) are essential in the proof of characterization of "hyperelliptic" in § 9.

In § 10, on a hyperelliptic Jacobian Jac C an analogy of Weierstrass'$\wp P$-function, which is also called the "$\wp$-function", is given; and the main point of this section is to discuss affine embedding of an open subset of Jac C by using P(z) and its derivatives, and to write down explicitly the equations defining the image variety. The final section § 11 of III a starts with rather long general discussion of the co-symplectic structure and presents KdV as a completely integrable dynamical system in an infinite dimensional space of pseudo-differential operators for the purpose of describing all the differential identities satisfied by hyperelliptic thetas. Actually the KdV dynamical system is solved by using $\wp$-function in § 10.

The second part III b of the chapter takes up general Jacobian theta functions (i.e., θ (z,Ω) for Ω the period matrix of an arbitrary Riemann surface C), except in § 5. The fundamental identity between such thetas is the ”trisecant” identity, due to *J. Fay* [”Theta functions on Riemann surfaces”, Lect. Notes Mat. 352 (1973; Zbl 0281.30013)], whose geometric interpretation tells us that the Kummer variety (corresponding to Jac C) has ∞^4 trisecants. This identity is proved in § 2 after a preliminary tool, the ”prime form” on $C \times C$, is discussed in § 1. Specializing the identity one gets several formulae involving derivatives of theta functions (§ 3) and constructs special solutions to many non-linear PDE's occuring in mathematical physics: the KP equation (in the general case) and KdV, Sine-Gordon (in the hyperelliptic case) in § 4. This part III b concludes with showing how to use the generalized Jacobian in the simplest case and its theta functions to describe and explain the soliton solutions to KdV as limits of the thetas. The contents of III c is already reviewed above.

At the end of the introduction, the author says there are two striking unsolved problems: (1) to find the differential identities in z satisfied by θ (z,Ω) for general Ω, and (2) the ”Schottky problem”; and gives four forthcoming papers as reference for the second problem. The reviewer refers to a new paper of *H. Morikawa* for the first problem [”A decomposition theorem on differential polynomials of theta functions”, Nagoya Math. J. 96, 113-124 (1984)]. Finally we mention that the book is stimulating to both young students and specialists.

Reviewer: S.Koizumi

MSC:

14K25	Theta functions and abelian varieties
14K30	Picard schemes, higher Jacobians
58J60	Relations of PDEs with special manifold structures (Riemannian, Finsler, etc.)
14-02	Research exposition (monographs, survey articles) pertaining to algebraic geometry
33E05	Elliptic functions and integrals
14H40	Jacobians, Prym varieties
58J15	Relations of PDEs on manifolds with hyperfunctions
35Q99	Partial differential equations of mathematical physics and other areas of application

Keywords:

Korteweg-de Vries dynamical system; theta functions; period matrices of Riemann surfaces; dynamical systems; effective divisors D; Jacobi coordinate; Jacobian variety; hyperelliptic curves; hyperelliptic thetas; Weierstrass'$\wp P$-function

Full Text: DOI